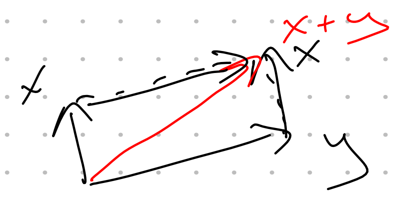
Properties of Vectors in $\mathbb{R}^n$

$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$



1) $x+y = y+x$

2) $(x+y)+z = x+(y+z)$

3) There is a vector called the zero vector, denoted 0, such that $x+0=x$ for all vectors x.

4) Let $-x$ denote the vector that has the same length as x, but the opposite direction. Then $x+(-x)=0$.

Scalar Multiplication

Let k be a real number and x be a vector.

kx is the vector whose magnitude is $|k|$ times the magnitude of x and whose direction is the same as x if $k>0$, and opposite to x if $k<0$. If $k=0$, $kx=0$.

e.g.

5) $1x = x$

6) $s(tx) = (st)x$ $\quad s,t \in \mathbb{R}$, x is an "arrow"

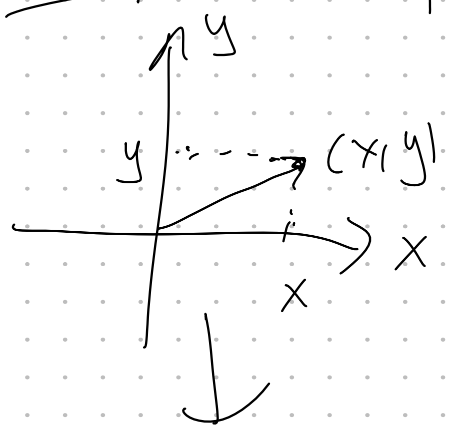
7) $r(x+y) = rx+ry$ $\quad r \in \mathbb{R}$, x,y are arrows

8) $(s+t)x = sx+tx$ $\quad s,t \in \mathbb{R}$, x is an arrow.

Note: we have *not* defined "multiplication of vectors"

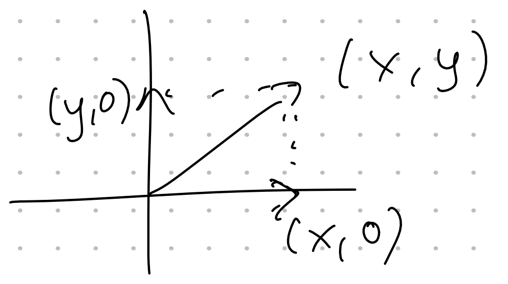
Notation: $(x,y) = \langle x,y \rangle$

Examples The plane $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$



algebraic versions of the geometric operations

$$\begin{cases} (x_1,y_1)+(x_2,y_2) = (x_1+x_2,\, y_1+y_2) \\ \text{for } k\in\mathbb{R} \quad k(x,y) = (kx,\, ky) \end{cases}$$



Thus, $(x,y) = (x,0)+(0,y)$

$\qquad = x(1,0)+y(0,1)$

Setting $i=(1,0)$ and $j=(0,1)$ we get $(x,y) = xi+yj$

The Zero Vector $\underset{\text{vector}}{0} = (\underset{\text{scalar}}{0,0})$

$-(x,y) = (-x,-y)$

Example 3-space $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

$(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2,\, y_1+y_2,\, z_1+z_2)$

$k(x,y,z) = (kx,ky,kz)$

The algebraic versions of vector operations naturally extend to n-tuples.

$\mathbb{R}^n = \{(x_1,x_2,\dots,x_n) \mid x_1,x_2,\dots,x_n \in \mathbb{R}\}$

$(x_1,x_2,\dots,x_n)+(y_1,y_2,y_3,\dots,y_n) = (x_1+y_1,\, x_2+y_2,\dots,x_n+y_n)$

$k(x_1,x_2,\dots,x_n) = (kx_1,kx_2,\dots,kx_n)$.

$0 = (0,0,\dots,0)$ $\qquad -(x_1,x_2,\dots,x_n) = (-x_1,-x_2,\dots,-x_n)$

the zero vector in $\mathbb{R}^n$

4.2 Definition of a Vector Space

Def^n Let V be a <u>non-empty set</u> (whose elements are called vectors) on which is defined an <u>addition</u> operation and <u>a scalar multiplication</u> operation with scalars in $\mathbb{F}$ ($=\mathbb{R}$ or $\mathbb{C}$). We call <u>V a vector space over $\mathbb{F}$</u>, provided the following ten conditions are satisfied:

A1. For all $u,v \in V$, $u+v$ is also in V. (Closure under addition)

A2. For all $v \in V$ and all $k \in \mathbb{F}$, $kv \in V$. (Closure under scalar multiplication)

A3 For all $u,v \in V$ $\quad u+v=v+u$ (Commutativity of vector addition)

A4 For all $u,v,w \in V$, $(u+v)+w = u+(v+w)$ $\quad$ associativity of vect. add.

A5. In V, there is a vector, denoted 0, satisfying $v+0=v$ for all $v\in V$. (Existence of a zero vector)

A6. For all $v \in V$, there is a vector, denoted $-v$, such that $v+(-v)=0$ (Existence of additive inverse)

A7. For all $v \in V$, $1v = v$ (unit property)

A8. For all $v \in V$, $r, s \in \mathbb{F}$ $(rs)v = r(sv)$ (associativity of scalar multiplication)

A9. For all $u, v \in V$ and all $r \in \mathbb{F}$ $r(u+v) = ru + rv$ (distributive property of scalar multiplication over vect. addition.)

A10. For all $v \in V$, $r, s \in \mathbb{F}$, $(r+s)v = rv + sv$ (distributive property of scalar mult. over scalar addition)

Remark: 1) To define a vector space one needs

a) A non-empty set of vectors V.

b) A set of scalars $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$)

c) An addition operation defined on V

d) A scalar multiplication defined on V.

Then all of the axioms A1–A10 should be satisfied.

2) Terminology: A vector space over the real numbers is called a <u>real vector space</u> and a vector space over the complex numbers is called a <u>complex vector space</u>.

Examples

1) $\mathbb{R}$ is a real vector space with the usual addition and multiplication.

2) $\mathbb{C}$ is a complex vector space with the usual addition and multiplication of complex numbers. $\mathbb{C}$ can also be though of as a real vector space by restricting the set of scalars to $\mathbb{R}$. kv $k \in \mathbb{F} = \mathbb{R}$, $v \in V = \mathbb{C}$

3) $\mathbb{R}^n$ (as defined earlier) is a real vector space.

4) The set of all 2×2 matrices with the usual operations of matrix addition and multiplication of a matrix by real numbers is a real vector space.

5) Let V be the set of all real-valued functions defined on an interval I.

$V = \{ f : I \to \mathbb{R} \}$

Define $f+g$ and kf for $f, g \in V$, $k \in \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$ for $x \in I$

$(kf)(x) = kf(x)$ for $x \in I$

Then V is a vector space (v.s)

Verification: Clearly A1 and A2 hold as $f+g$ and kf are both real valued functions on I.

A3. $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$ for all $x \in I$.

A4. $[(f+g)+h](x) = (f+g)(x) + h(x)$

$= (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$

$= f(x) + (g+h)(x) = [f + (g+h)](x)$

A5. Define the zero function by $\underset{\substack{\uparrow \\ \text{function}}}{0(x)} = \underset{\substack{\uparrow \\ \text{scalar}}}{0}$

then $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$ ✓

A6. If $f \in V$ define $-f$ by $(-f)(x) = -f(x)$

then $(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x)$

So $f + (-f) = 0$

A7 $(1f)(x) = 1(f(x)) = f(x)$ ✓

A8 let $f \in V$, $r, s \in \mathbb{R}$ Then

$[(rs)f](x) = (rs)f(x) = r(sf(x))$

$= r((sf)(x)) = (r(sf))(x)$

A9. Let $f, g \in V$, $r \in \mathbb{R}$

$[r(f+g)](x) = r((f+g)(x)) = r(f(x) + g(x)) = rf(x) + rg(x) = (rf)(x) + (rg)(x)$

$= ((rf) + (rg))(x)$ ✓

A10. $f \in V$ $r, s \in \mathbb{R}$ Then

$((r+s)f)(x) = (r+s)f(x) = rf(x) + sf(x) = (rf)x + (sf)(x) = ((rf) + (sf))(x)$ ✓

So V is a real vector space.