Properties of Determinants

P7) From (P3, 4, 6), it follows that ERO (elementary row op.) do not change whether det is 0 or non-zero.

Recall $A_{n\times n}$ is invertible $\iff$ $\text{rank}(A) = n \iff$ RREF of A is I_n $\Rightarrow \det(A) \neq 0$

A is not invertible $\iff \text{rank}(A) < n \iff$ RREF of A has a zero row $\Rightarrow \det(A) = 0$

Thus $\det(A) \neq 0$ if and only if A is invertible.

P8) Any scalar quantity $Q(A)$ having these properties is indeed $\det(A)$. ($Q(A)$ must be $= \det A$)

Thm $\det(AB) = \det(A)\det(B)$

In particular, $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det I_n = 1$

So $\det(A^{-1}) = \dfrac{1}{\det(A)}$

3.3 The Cofactor Expansions.

Def^n Let A be an $n\times n$ matrix. The (i,j)th minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

e.g. let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad M_{11} = \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} = -3$$

$$M_{22} = \det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$$

$$M_{13} = \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} = -3$$

Def^n let A be an $n\times n$ matrix. The (i,j)th cofactor C_{ij} of the element a_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

e.g.
$$A = \begin{pmatrix} \cancel{1} & \cancel{0} & \cancel{2} \\ 3 & \cancel{6} & 5 \\ 1 & \cancel{2} & 0 \end{pmatrix} \qquad M_{12} = \det\begin{pmatrix} 3 & 5 \\ 1 & 0 \end{pmatrix} = -5$$

$$C_{12} = (-1)^{1+2}(-5) = 5$$

$$M_{33} = \det\begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix} = 6 \qquad C_{33} = (-1)^{3+3}(6) = 6$$

signs alternate like this
$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thm (Cofactor Expansion Theorem) let A be an $n\times n$ matrix. If we multiply the elements in any row (or column) of A by their cofactors, then the sum of the resulting product is $\det(A)$. Thus,

1) If we expand along row i;
$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}$$

2) If we expand along column j;
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}$$

e.g.
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M_{11} = \det(d) = d \qquad M_{12} = \det(c) = c$$

$$C_{11} = M_{11} = d \qquad C_{12} = -M_{12} = -c$$

$$\det(A) = aC_{11} + bC_{12} = ad - bc$$

e.g.
$$A = \begin{pmatrix} 1 & 5 & 2 \\ 3 & 7 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

let's expand along the first row.

$$\det(A) = 1\begin{vmatrix} 7 & 1 \\ 0 & 4 \end{vmatrix} - 5\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 3 & 7 \\ 0 & 0 \end{vmatrix} = 1(28) - 5(12) + 2(0) = -32$$

$\boxed{\text{Notation: } \det(A) = |A|}$

Cofactor expansion along row 3 gives us;
$$\det(A) = 0 - 0 + 4\begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = 4(7-15) = -32$$

e.g. Compute $\det(A)$ for
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ -1 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 6 & 5 \end{pmatrix} = B$$

$\left.\begin{array}{l} -2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array}\right\}$ these operations don't change det!

Cofactor expand along the first column.

$$\det(A) = \det(B) = 1\begin{vmatrix} 1 & -1 \\ 6 & 5 \end{vmatrix} = 5 + 6 = 11$$

Remark: $\det(A^T) = \det(A)$ because the i^{th} column cofactor expansion of A^T is the i^{th} row cofactor expansion of A.

The adjoint method for A^{-1}

Def^n If every element in an $n \times n$ matrix A is replaced by its cofactor, the resulting matrix is called the matrix of cofactors and is denoted M_C.

The transpose of M_C is called the adjoint of A and is denoted $\operatorname{adj}(A)$.

Thus the elements of $\operatorname{adj}(A)$ are $\operatorname{adj}(A)_{ij} = C_{ji}$

$$\left(\operatorname{adj}(A) = M_C^T \right)$$

e.g. Determine $\operatorname{adj}(A)$ for

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 2 \end{pmatrix}$$

$M_{11} = \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 2 \quad M_{12} = \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} = -11 \quad M_{13} = -5$

$M_{21} = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = 2 \quad M_{23} = 10$

$M_{31} = \begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} = -6 \quad M_{32} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3 \quad M_{33} = 5$

$$M_C = \begin{pmatrix} 2 & 11 & -5 \\ 4 & 2 & -10 \\ -6 & -3 & 5 \end{pmatrix} \leftarrow \text{cofactors}$$

So $\operatorname{adj}(A) = M_C^T = \begin{pmatrix} 2 & 4 & -6 \\ 11 & 2 & -3 \\ -5 & -10 & 5 \end{pmatrix}$.

Thm (The adjoint method for A^{-1})

If $\det(A) \neq 0$ then $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

Continuing from the prev example,

$$\det(A) = 5 \begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} + 0 + 2 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -30 + 10 = -20$$

So $A^{-1} = -\dfrac{1}{20} \begin{pmatrix} 2 & 4 & -6 \\ 11 & 2 & -3 \\ -5 & -10 & 5 \end{pmatrix}$

Cramer's rule

Given an invertible $n \times n$ matrix A and the vector equation $Ax = b$

let B_k denote the matrix obtained by replacing the k^{th} column of A with b.

So,

$$B_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & & b_2 & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & & b_n & & a_{nn} \end{pmatrix}$$

Thm If $\det(A) \neq 0$ the unique solution to the $n \times n$ system $Ax = b$ is $x = (x_1, x_2, \ldots, x_n)$ where

$$x_k = \frac{\det(B_k)}{\det(A)} \quad k = 1, 2, \ldots, n$$

Example Use Cramer's rule to solve the system

$$3x_1 - x_2 = 1$$
$$-5x_1 + 2x_2 = 4$$

$A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \quad \det(A) = 6 - 5 = 1$

$\det(B_1) = \begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} = 2 + 4 = 6$

$\det(B_2) = \begin{vmatrix} 3 & 1 \\ -5 & 4 \end{vmatrix} = 12 + 5 = 17$

So $x_1 = \dfrac{\det B_1}{\det A} = \dfrac{6}{1} = 6$

$x_2 = \dfrac{\det B_2}{\det A} = \dfrac{17}{1} = 17$

$x = (6, 17)$ is the unique solution.

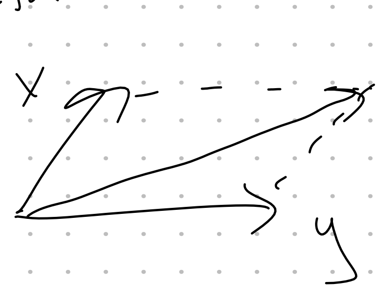
4. Vector Spaces

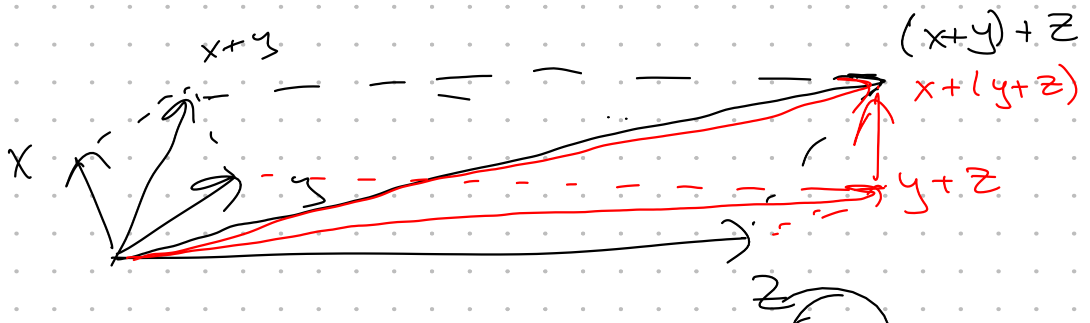
4.1 Vectors in the Euclidean Space $(\mathbb{R}^n)$

Vector are "arrows". They have a magnitude (length) and a direction.

Vector addition

① $x + y = y + x$ (commutative)





So we have associativity. ②

③ There is a vector called the zero vector, denoted 0, such that $x + 0 = x$ for all vectors x. $0 = \cdot$