Last time: $\det(a) = a$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

"if and only if"

$\det(A) \neq 0 \iff A$ is invertible

General Case (det of $n\times n$ matrix)

Permutations: Rearrangements of integers $1, 2, \dots, n$

Example All permutations of $1,2,3$:

$(1,2,3)$ $(1,3,2)$ $(2,1,3)$ $(2,3,1)$

$(3,1,2)$ $(3,2,1)$

$(1,2,3,\dots,n)$ is said to be in its natural increasing order.

For $i<j$ p_i and p_j are said to be inverted if $p_i > p_j$.

We call (p_i, p_j) an inversion.

$N(p_1, p_2, \dots, p_n)$ = the total number of inversions.

Def^n If $N(p_1, \dots, p_n)$ is even (odd) then $(p_1, \dots, p_n)$ is called an even permutation (odd permutation). We also say that $(p_1, \dots, p_n)$ has even (odd) parity.

Define $\sigma(p_1,\dots,p_n) = \begin{cases} +1 & \text{if } (p_1,\dots,p_n) \text{ is even} \\ -1 & \text{if } (p_1,\dots,p_n) \text{ is odd} \end{cases}$

called the sign of $(p_1, p_2, \dots, p_n)$

Def^n (Determinant)

Let $A = (a_{ij})$ be an $n\times n$ matrix.
The determinant of A, denoted $\det(A)$, is defined as follows:

$$\det(A) = \sum \sigma(p_1, p_2, \dots, p_n)\, a_{1p_1} a_{2p_2} \cdots a_{np_n}$$

where the summation is over the $n!$ distinct permutations.
(Not very practical!)

Property #1

P1) If A has a row (or column) of zeros, then $\det(A) = 0$
(In particular, $\det(0) = 0$)

P3) det is additive in every row when the other rows are fixed.

$$A = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ a+b \\ \vdots \\ r_n \end{pmatrix}$$ where $r_1, r_2, \dots, r_n, a, b$ are rows.

Webwork quadratic polynomial $p(x)$

$p(0) = 0$ $p(1) = a$ $p(2) = b$

$p(x) = Ax^2 + Bx + C$

$$\begin{aligned} p(0) &= \boxed{0 + 0 + C = 0} \\ p(1) &= A + B + C = a \\ p(2) &= 4A + 2B + C = b \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & a \\ 4 & 2 & 1 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 4 & 2 & 1 & b \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{\overset{-4R_1+R_2 \to R_2}{}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -3 & b-4a \\ 0 & 0 & 1 & 0 \end{array}\right) \underset{\sim}{\overset{-\frac{1}{2}R_2 \to R_2}{}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3/2 & \frac{b-4a}{-2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

A B C

e.g. $N(1,3,2) = 1$ as $(3,2)$ is the only inversion

$N(4,1,3,5,2) = 5$ as $(4,1)$ $(4,3)$ $(4,2)$ $(3,2)$, $(5,2)$ are all the inversions

so $(1,3,2)$ is an odd permutation

$(4,1,3,5,2)$ is an // //

$N(3,1,2) = 2$ so $(3,1,2)$ has even parity

$N(1,2,3) = 0$ so $(1,2,3)$ has even parity.

$\det(a) = \overset{(1)}{a}$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \overset{(1,2)}{+ad} \overset{(2,1)}{- bc}$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi$$

3.2 Properties of Determinants

Thm If A is $n\times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$$

Pf

$$\begin{pmatrix} a_{11} & * & * & \cdots & * \\ 0 & a_{22} & * & & \vdots \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & a_{n-1,n-1} & * \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

P2) I_n is in particular an upper (also lower) triangular matrix so by the theorem above $\det(I_n) = 1 \cdot 1 \cdots 1 = 1$

then

$$\det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ a+b \\ \vdots \\ r_n \end{pmatrix} = \det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ a \\ \vdots \\ r_n \end{pmatrix} + \det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ b \\ \vdots \\ r_n \end{pmatrix}$$

Similarly, $\det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ cb \\ \vdots \\ r_n\end{pmatrix} = c\det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ b \\ \vdots \\ r_n\end{pmatrix}$.

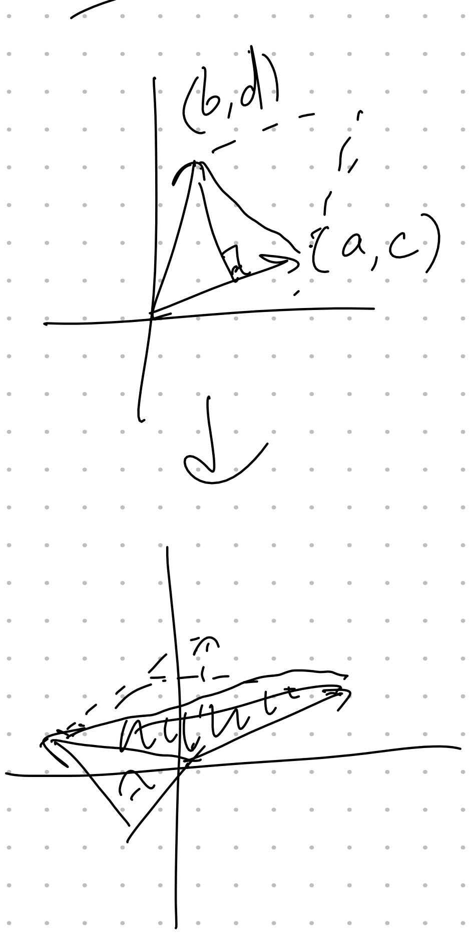
Moreover since $cA = \begin{pmatrix} cr_1 \\ cr_2 \\ \vdots \\ cr_n\end{pmatrix}$,

$$\det(cA) = c^n \det(A)$$

We also have the analogous properties for columns.

P4) If B is the matrix obtained by permuting two rows of A then $\det(B) = -\det A$

Geometric: P6 (for $n=2$) says



$\begin{pmatrix} a & c \\ b & d\end{pmatrix}$

$\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}$

= area of the parallelogram spanned by $\begin{pmatrix} a \\ b\end{pmatrix}\begin{pmatrix} c \\ d\end{pmatrix}$

P5) If there are two identical rows, $\det(A) = 0$ (Since by P4, $\det(A) = -\det(A)$)

P6) If B is obtained from A by $kR_i + R_j \to R_j$ then $\det B = \det A$

$A = \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_n\end{pmatrix}$ $B = \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ kr_i + r_j \\ \vdots \\ r_n\end{pmatrix}$

$\det(B) \underset{(P3)}{=} \det\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ kr_i \\ \vdots \\ r_n\end{pmatrix} + \det\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_n\end{pmatrix}$

$= k\det\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_i \\ \vdots \\ r_n\end{pmatrix} + \det A \underset{(P5)}{=} 0 + \det A = \det A$