2.5 Gaussian Elimination

Solving a system of linear equations by reducing its augmented matrix $A^\#$ to a row-echelon form and back substituting is called Gaussian elimination. Reducing $A^\#$ to RREF and solving the system is called Gauss-Jordan elimination.

Example Solve by Gaussian elimination and Gauss-Jordan elimination

$$2x_1 + x_2 = 5$$
$$x_1 + 7x_2 = 9$$

$$A^\# = \left(\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 7 & 9 \end{array}\right) \overset{1}{\sim} \left(\begin{array}{cc|c} 1 & 7 & 9 \\ 2 & 1 & 5 \end{array}\right)$$

(pivot column, pivot position)

$$\overset{2}{\sim} \left(\begin{array}{cc|c} 1 & 7 & 9 \\ 0 & -13 & -13 \end{array}\right)$$

pivot column = pivot position because in this case it is 1×1

1) $R_1 \leftrightarrow R_2$
2) $-2R_1 + R_2 \to R_2$
3) $-\frac{1}{13}R_2 \to R_2$

$$\overset{3}{\sim} \left(\begin{array}{cc|c} 1 & 7 & 9 \\ 0 & 1 & 1 \end{array}\right) \rightsquigarrow \begin{aligned} x_1 + 7x_2 &= 9 \\ x_2 &= 1 \end{aligned}$$

So $x_2 = 1$ and $x_1 + 7 = 9$ thus, $x_1 = 2$. So $(x_1, x_2) = (2,1)$ is the only solution. Next, we continue to find RREF to use Gauss-Jordan elimination.

$2x = 6$

~~$0 = 0$~~

4) $-7R_2 + R_1 \to R_1$

$$\overset{4}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right) \rightsquigarrow \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned}$$

So $(x_1, x_2) = (2,1)$ is the only solution. We obtained this by Gauss-Jordan.

Lemma Consider the $m \times n$ linear system $Ax = b$. Let $A^\#$ denote the augmented matrix. If $\operatorname{rank}(A) = \operatorname{rank}(A^\#) = n$ then the system has a unique solution.

"Proof"

$$A^\# \sim \left(\begin{array}{cccccc|c} 1 & * & * & & & * & * \\ 0 & 1 & * & & & & \vdots \\ \vdots & 0 & 1 & & & * & \\ \vdots & & & \ddots & 1 & * & * \\ 0 & & \cdots & & 0 & 1 & * \\ 0 & 0 & \cdots & & & 0 & 0 \\ \vdots & & & & & & \\ 0 & \cdots & & & & 0 & 0 \end{array}\right) \rightarrow \begin{aligned} x_1 + *x_2 + \cdots + *x_n &= * \\ &\vdots \\ x_{n-1} + *x_n &= * \\ x_n &= * \end{aligned}$$

So by back substitution we see that there is a unique solution.

Lemma If $\operatorname{rank}(A) < \operatorname{rank}(A^\#)$ the system is inconsistent.

"Pf"

$$A^\# = (A|b) \sim \left(\begin{array}{cccc|c} 1 & * & & * & \vdots \\ & 1 & & & \\ 0 & & \ddots\ 1 & * & * \\ & & & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \\ & & & & 0 \\ 0 & & & 0 & 0 \end{array}\right) \rightarrow 0x_1 + 0x_2 + \cdots + 0x_n = 1$$

this is $0 = 1$ so it won't have a solution no matter what x_i's are.

Example Solve

$$2x_1 + x_2 = 5$$
$$-x_1 - \tfrac{1}{2}x_2 = 3$$

$$A^\# = \left(\begin{array}{cc|c} 2 & 1 & 5 \\ -1 & -\frac{1}{2} & 3 \end{array}\right) \overset{\frac{1}{2}R_1 \to R_1}{\sim} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ -1 & -\frac{1}{2} & 3 \end{array}\right)$$

(the coefficient part is labeled A)

$$\overset{R_1 + R_2 \to R_2}{\sim} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & \frac{11}{2} \end{array}\right)$$

$$1x_1 + \tfrac{1}{2}x_2 = \tfrac{5}{2}$$
$$0x_1 + 0x_2 = \tfrac{11}{2} \leftarrow \text{so there are no solutions}$$

$$\underbrace{\operatorname{rank}\begin{pmatrix} 2 & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}}_{\operatorname{rank}(A)} = \operatorname{rank}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} = 1 < \operatorname{rank}(A^\#) = \operatorname{rank}\left(\begin{array}{cc|c} 2 & 1 & 5 \\ -1 & -\frac{1}{2} & 3 \end{array}\right)$$
$$= \operatorname{rank}\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 \end{array}\right) = 2$$

Example Solve

$$x_1 + 2x_2 - x_3 = 4$$
$$4x_1 - x_2 + 5x_3 = 0$$
$$3x_1 - 3x_2 + 6x_3 = -4$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 4 & -1 & 5 & 0 \\ 3 & -3 & 6 & -4 \end{array}\right)$$

(pivot)

$$\overset{1}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -9 & 9 & -16 \\ 0 & -9 & 9 & -16 \end{array}\right) \overset{2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 16/9 \\ 0 & -9 & 9 & -16 \end{array}\right) \overset{3}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 16/9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(pivot)

1) $-4R_1 + R_2 \to R_2$
$-3R_1 + R_3 \to R_3$
2) $-\frac{1}{9}R_2 \to R_2$
3) $9R_2 + R_3 \to R_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ x_2 - x_3 &= 16/9 \end{aligned}$$

So note that for any value of x_3, this system will have a unique solution

$$\begin{aligned} x_1 + 2x_2 &= 4 + x_3 \\ x_2 &= 16/9 + x_3 \end{aligned}$$

So set $x_3 = t \in \mathbb{R}$ then any solution to the system $Ax=b$ will be of the form

$$\begin{aligned} x_1 + 2x_2 &= 4 + t \\ x_2 &= \tfrac{16}{9} + t \end{aligned}$$

So $x_1 + 2\left(\frac{16}{9} + t\right) = 4 + t$

$x_1 = 4 + t - \frac{32}{9} - 2t = \frac{4}{9} - t$. So $(x_1, x_2, x_3) = \left(\frac{4}{9} - t, \frac{16}{9} + t, t\right)$

So the solution set is

$S = \{ (\frac{4}{9} - t, \frac{16}{9} + t, t) \mid t \in \mathbb{R} \}$

Here x_3 is called a <u>free variable</u>. x_1 and x_2 are called <u>bound variables</u>.

$$A^\# \sim \left(\begin{array}{ccc|c} \textcircled{1} & 2 & -1 & 4 \\ 0 & \textcircled{1} & -1 & 16/9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_1 x_2 x_3

x_1 and x_2 corresponds to columns with leading 1s so they are <u>bound variables</u>.

The column that corresponds to x_3 <u>does not</u> have a leading 1 so it is a <u>free variable</u>.

Note that $(\frac{4}{9} - t, \frac{16}{9} + t, t) = (\frac{4}{9}, \frac{16}{9}, 0) + t(-1, 1, 1)$ so geometrically this is a line.

Note that homogeneous systems ($Ax=b$ for $b=0$) always have at least one solution $x=0$ called <u>the trivial solution</u>.

<u>Example</u> Determine the solution set to $Ax=0$ if

$$A = \begin{pmatrix} 2 & 0 & 3 & -5 \\ 1 & 0 & -1 & 0 \\ 3 & 0 & 7 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 3 & -5 \\ 3 & 0 & 7 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 10 & -10 \end{pmatrix} \sim \begin{pmatrix} \textcircled{1} & 0 & -1 & 0 \\ 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_1 x_2 x_3 x_4 — bound / free

$$\begin{aligned} x_1 + 0x_2 - x_3 + 0x_4 &= 0 \\ x_3 - x_4 &= 0 \end{aligned} \rightsquigarrow \begin{aligned} x_1 - x_3 &= 0 \\ x_3 - x_4 &= 0 \end{aligned}$$

Set $x_4 = t$ $x_2 = s$

then $x_3 = x_4 = t$ $x_1 = x_3 = t$

$(x_1, x_2, x_3, x_4) = (t, s, t, t)$ So $S = \{(t, s, t, t) \mid t, s \in \mathbb{R}\}$

$= t(1, 0, 1, 1) + s(0, 1, 0, 0)$ so Geometrically, S is a plane!

2.6 The Inverses of a Square Matrix.

For a given $n \times n$ matrix A, is there an $n \times n$ matrix B s.t.

$AB = I_n$ and $BA = I_n$?