Last time:

$$(*)\begin{cases} \dfrac{dx_1}{dt} = 5t\,x_1 + e^t x_2 + \sin t \\[2mm] \dfrac{dx_2}{dt} = 0\,x_1 - 3\ln(t)\,x_2 + t^3 \end{cases}$$

Set $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

$$A(t) = \begin{pmatrix} 5t & e^t \\ 0 & -3\ln t \end{pmatrix} \quad b(t) = \begin{pmatrix} \sin t \\ t^3 \end{pmatrix}$$

$$\overset{(*)}{x' = A(t)x + b(t)}$$

First Midterm: 2/20 Thursday 8-9:20 am
Hubbell Auditorium.

2.4 Row-echelon Matrices and Elementary row operations

Solve
$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - 4x_3 &= -4 \\ x_3 &= 3 \end{aligned}\right\} (**)$$

Very easy using back substitution!

$x_3 = 3$ So $x_2 - 4(3) = -4$

$x_2 = 8$ So $x_1 + 8 + 3 = 6$

$x_1 = -5$

So if we can simplify a complicated system to a system like $(**)$, we are done. Note the augmented matrix that corresponds to $(**)$ is

$$A^\# = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Def^n An $m \times n$ matrix is called a row-echelon matrix if it satisfies the following 3 conditions

1) If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.
2) The first non-zero entry in any non zero row is a 1 (called a leading 1)
3) The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

Examples

$$\begin{pmatrix} 1 & -8 & -3 & 7 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

non-example

$$\cancel{\begin{pmatrix} 1 & 5 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix}}$$

non-examples

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & \cancel{2} & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 4 \end{pmatrix}$$

Elementary Row Operations

1) Permute equations $\Longleftrightarrow$ Permute Rows
2) Multiply an equation by a non-zero const
$\Updownarrow$
Multiply a row by a non-zero const
3) Add a multiple of one eq. to another eq.
$\Updownarrow$
Add a multiple of one row to another row

$$\begin{aligned} x + y &= 2 \\ + \quad x - y &= 0 \\ \hline 2x + 0y &= 2 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 0 & 2 \end{array}\right)$$

Notations: R_i = is the i^{th} row

1) $R_i \Longleftrightarrow R_j$
2) $kR_i \to R_i$
3) $kR_i + R_j \to R_j$

Also $A \sim B$ if B is obtained from A by a sequence of these operations.

e.g.

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 8 \\ 1 & 5 & 3 & 2 \end{array}\right) \overset{1}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 5 & 3 & 2 \end{array}\right)$$

$$\overset{2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 3 & 0 & -2 \end{array}\right) \qquad 1)\ \tfrac{1}{2}R_1 \to R_1$$

$$\qquad 2)\ -R_1 + R_2 \to R_2$$

$$\overset{3}{\sim} \left(\begin{array}{ccc|c} 0 & 3 & 0 & -2 \\ 1 & 2 & 3 & 4 \end{array}\right) \qquad 3)\ R_1 \Longleftrightarrow R_2$$

We say A is row-equivalent to B if $A \sim B$.

Theorem If $A^\# \sim B^\#$ then two systems have the same set of solutions.

Theorem Every matrix is row-equivalent to a row echelon matrix.

Example Reduce $A = \begin{pmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{pmatrix}$ to row-echelon form.

$$A \overset{1}{\sim} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -5 & 2 \\ 1 & 0 & -3 & 4 \end{pmatrix} \overset{2}{\sim} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & -1 & -2 & 3 \end{pmatrix}$$

1) $R_1 \leftrightarrow R_2$
2) $-3R_1+R_2 \to R_2$
$-R_1+R_3 \to R_3$
3) $-R_2 \to R_2$
4) $R_2+R_3 \to R_3$
5) $\frac{1}{4}R_3 \to R_3$

$$\overset{3}{\sim}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \end{pmatrix} \overset{4}{\sim}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \overset{5}{\sim}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Def^n The number of non-zero rows in any row-echelon form of a matrix A is called the rank of A and is denoted $\text{rank}(A)$

e.g. $A=\begin{pmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\text{rank}(A)=3.$

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \overset{-2R_1+R_2\to R_2}{\sim} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

So $\text{rank}\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}=1$

Def^n An $m\times n$ matrix is called a <u>reduced row-echelon</u> matrix if it satisfies the following:

1) It is a row-echelon matrix.
2) Any column that contains a leading 1 has zeros everywhere else.

Examples

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

non-Examples

$$\begin{pmatrix} 1 & 3 & 2 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 & \overset{0}{\cancel{7}} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & \cancel{2} & 4 \end{pmatrix}$$

Thm Any $m\times n$ matrix has a unique reduced row-echelon form (RREF).

Example Determine the RREF of

$$A=\begin{pmatrix} 3 & -7 & -5 \\ 1 & -3 & -3 \\ 2 & -2 & 2 \end{pmatrix} \overset{1}{\sim} \begin{pmatrix} 1 & -3 & -3 \\ 3 & -7 & -5 \\ 2 & -2 & 2 \end{pmatrix}$$

1) $R_1 \leftrightarrow R_2$
2) $-3R_1+R_2 \to R_2$
$-2R_1+R_3 \to R_3$
3) $\frac{1}{2}R_2 \to R_2$
$\frac{1}{4}R_3 \to R_3$
4) $-R_2+R_3 \to R_3$
5) $3R_2+R_1 \to R_1$

$$\overset{2}{\sim}\begin{pmatrix} 1 & -3 & -3 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix} \overset{3}{\sim}\begin{pmatrix} 1 & -3 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \overset{4}{\sim}\begin{pmatrix} 1 & -3 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(row-echelon but not RREF)

$$\overset{5}{\sim}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(So $\text{rank}(A)=2$)