Last time: dot products:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$(a_1 \cdots a_n) \cdot (b_1 \cdots b_n) = a_1 b_1 + \cdots + a_n b_n$$

Matrix product of a row vec. and a col. vec.

$$\underset{1\times n}{(a_1 \cdots a_n)} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}_{n\times 1} = (a_1 b_1 + \cdots + a_n b_n)_{1\times 1}$$

Def^n If A is an $\underline{m\times n}$ matrix and B is an $\underline{n\times p}$ matrix, then AB is the $m\times p$ matrix whose (i,j)-entry is the number we get when we multiply the ith row of A with jth column of B.

e.g.

$$\begin{pmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{pmatrix}_{2\times 3} \begin{pmatrix} -4 & 1 \\ 3 & -1 \\ -9 & -2 \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} 8+3-27 & -2-1-6 \\ -16-6-54 & 4+2-12 \end{pmatrix}_{2\times 2} = \begin{pmatrix} -16 & -9 \\ -76 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0-2+3+8 \\ 0-6+7+16 \end{pmatrix} = \begin{pmatrix} 9 \\ 17 \end{pmatrix}$$

$2 \times \cancel{4}$ $\quad \cancel{4} \times 1$ $\qquad 2\times 1$

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} (2\ 5\ 7) = \begin{pmatrix} 6 & 15 & 21 \\ 2 & 5 & 7 \\ 4 & 10 & 14 \end{pmatrix}_{3\times 3}$$

$3\times \cancel{1}$ $\quad \cancel{1}\times 3$

$$(2\ 5\ 7) \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = (6+5+14) = (25)$$

$1\times\cancel{3}$ $\quad \cancel{3}\times 1$ $\qquad 1\times 1$

let $A=(a_{ij})$ be an $m\times n$ matrix,
$B=(b_{ij})$ be an $n\times p$ matrix

and $C = AB$ then

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \text{for } 1 \le i \le m,\ 1 \le j \le p.$$

This is called the index form of the matrix product.

Note $A_{m\times n} B_{n\times p} = C_{m\times p}$

same

result

Properties:

- $A(BC) = (AB)C$ (associativity)
- $A(B+C) = AB + AC$ (left distributivity)
- $(A+B)C = AC + BC$ (right //)

Next we consider if $AB = BA$ in general.

Examples 1) $A = (5\ 4)_{1\times 2}$ $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}_{2\times 3}$

$AB = (5\ \ 14\ \ 31)$

BA is not even defined!

2) $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $B = (5\ 6)$

$AB = \begin{pmatrix} 5 & 6 \\ 10 & 12 \end{pmatrix} \neq BA = (17)$

3) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq BA = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

So in general the matrix multiplication is not commutative. There are special cases where $AB = BA$

Notation: $A^2 = AA$ or $A^3 = AAA$ etc (for a square matrix A)

Example $(A+B)^2 = (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

Notation $I_n = \text{diag}(1,1,1,\ldots,1)$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 1\ \ 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Def^n The elements of I_n can be represented by the Kronecker delta symbol δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then $I_n = (\delta_{ij})$

Properties $A_{m\times n} I_n = A_{m\times n}$

$I_m A_{m\times n} = A_{m\times n}$

Properties of transpose

1) $(A^T)^T = A$

2) $(A+C)^T = A^T + C^T$

3) $\boxed{(AB)^T = B^T A^T}$

$$\left(A_{m\times n}\; B_{n\times p}\right)^T = \left(\;{}_{m\times p}\right)^T = {}_{p\times m}$$

$A^T_{n\times m} \quad B^T_{p\times n}$

Everything above applies to matrix functions as well. Furthermore, we may scalar multiply by <u>scalar functions</u>.

e.g. $$\underbrace{e^t\begin{pmatrix} 1 & \ln t \\ t^2 & \frac{1}{t}\end{pmatrix}}_{A(t)} = \begin{pmatrix} e^t & e^t \ln t \\ e^t t^2 & \frac{e^t}{t}\end{pmatrix}$$

$$\frac{dA}{dt} = \begin{pmatrix} 0 & \frac{1}{t} \\ 2t & -t^{-2}\end{pmatrix} \quad \text{or integrate}$$

$$\int_0^1 \begin{pmatrix} 2 & 2t \\ e^t & 6t^2\end{pmatrix} dt = \left.\begin{pmatrix} 2t & t^2 \\ e^t & 2t^3\end{pmatrix}\right|_0^1$$

$$= \begin{pmatrix} 2 & 1 \\ e & 2\end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} = \begin{pmatrix} 2 & 1 \\ e-1 & 2\end{pmatrix}$$

2.3 Terminology for Systems of Linear Equations

e.g. $$\left\{\begin{array}{l} 2x + 3y = 5 \\ -x + 5y = 0 \end{array}\right\}$$

<u>Defn</u> The general $m\times n$ system of linear equations is of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} (*)$$

where the <u>system coefficients</u> a_{ij} and the <u>system constants</u> b_i are given scalars

and $x_1, x_2, \ldots, x_n$ denote the unknowns in the system.

If $b_i = 0$ for all i, then the system is called <u>homogeneous</u>; otherwise it is called <u>non-homogeneous</u>.

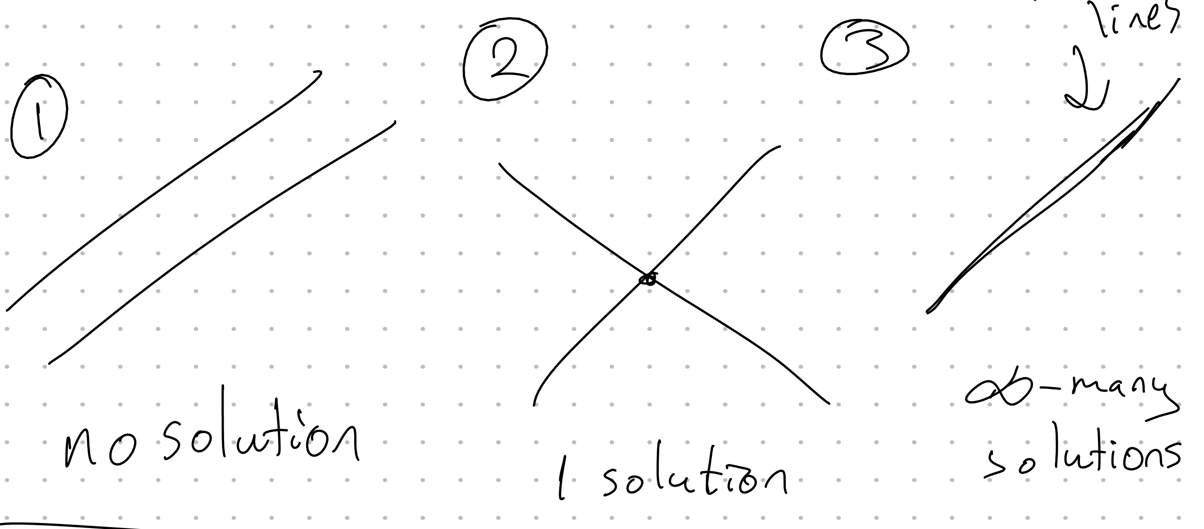
By a solution to the system $(*)$ we mean an ordered n-tuple of scalars $(c_1, c_2, \ldots, c_n)$ which when substituted for $x_1, x_2, \ldots x_n$ into the LHS of $(*)$, yield the values on the RHS. The set of all solutions to the system $(*)$ is called the solution set to the system.

<u>Example</u> $$x_1 + 2x_2 + 3x_3 + 4x_4 = 8$$
$$2x_1 + 5x_2 + 10x_3 + 5x_4 = 8$$

<u>check</u> that $(24, -8, 0, 0)$ is a solution of the system.

$\boxed{\text{Question}}$ Does $(*)$ have solutions? If so, how many?

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$



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the same is true for an arbitrary system, namely there is either no solution, exactly 1 solution, or ∞-many solutions.

<u>Defn</u> A system of eqs that has <u>at least</u> one solution is said to be <u>consistent</u>, whereas a system that has no solution is called <u>inconsistent</u>.

Note that the system $(*)$ can be written as the following matrix multiplication

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m\end{pmatrix}$$

the matrix A is called the matrix of coefficients of $(*)$.

We set

$$A^{\#} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$A^{\#}$ is called the augmented matrix. This contains all the information!

Further we define

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m\end{pmatrix}$$

So $Ax = b$ is the vector equation corresponding to ⊛

x : the vector of unknowns

b : the right hand side vector

We may also use such notation for diff. eq.s

e.g. $\dfrac{dx_1}{dt} = 5t\,x_1 + e^t x_2 + \sin t$

$\dfrac{dx_2}{dt} = 0\,x_1 - 3\ln(t)\,x_2 + t^3$

set $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

$$\frac{dx}{dt} = \begin{pmatrix} 5t & e^t \\ 0 & -3\ln t \end{pmatrix} x + \begin{pmatrix} \sin t \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} 5t & e^t \\ 0 & -3\ln t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5t\,x_1 + e^t x_2 \\ 0\,x_1 - 3\ln t\,x_2 \end{pmatrix}$$