Chp 2 Matrices and System of Linear Equations

2.1: Matrices: Definition and Notation

Def^n An $m \times n$ (read "m by n") **matrix** is a rectangular array of numbers arranged in m horizontal rows and n vertical columns.

e.g.
1st row

$$A = \begin{pmatrix} 9 & 3 & -2 \\ 10 & 0 & -1 \\ -5 & 7 & 4 \end{pmatrix}$$ is a 3×3 matrix

3rd col

$$B = \begin{pmatrix} -1 & 0 \\ -5 & 2 \\ 7 & 14 \\ 2 & 1 \end{pmatrix}$$ is a 4×2 matrix

The expression $m \times n$ is called the size of the matrix.

The entry in the i^{th} row and j^{th} column of a matrix A is denoted by a_{ij}.

e.g. $a_{13} = -2 \quad a_{11} = 9$

$b_{42} = 1$

Two matrices A and B are equal written $A = B$ if they have the same size and $a_{ij} = b_{ij}$ for all possible values of i and j.

e.g. $\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1+1 \end{pmatrix}$

Def^n A $1 \times n$ matrix is called a row n-vector

An $n \times 1$ matrix is called a column n-vector

The entries of a row or a column n-vector are called the components of the vector.

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e.g. $a = \begin{pmatrix} -2 & \frac{1}{3} & 5 \end{pmatrix}$ is a row 3-vector

$b = \begin{pmatrix} 3 \\ 0 \\ -1 \\ -1 \end{pmatrix}$ is column 4-vector

$a_1 = -2 \quad a_2 = \frac{1}{3} \quad a_3 = 5$

$b_2 = 0$

If $a_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$

then $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

similarly if $b_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $b_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $b_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Def^n If we interchange the rows and columns in an $m \times n$ matrix A, we obtain an $n \times m$ matrix called the transpose of A, denoted A^T.

In index notation

$a^T_{ij} = a_{ji}$ $\quad A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 2 \end{pmatrix}$

$B = \begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}, \quad B^T = \begin{pmatrix} 5 & -1 \\ 0 & 2 \end{pmatrix}$

Def^n An $n \times n$ matrix is called a square matrix. If A is a square matrix, the entries a_{ii} are called the diagonal entries.

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e.g. $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ Diagonal entries

The sum of all diagonal entries of an $n \times n$ matrix A is called the trace of A, denoted by $tr(A)$.

$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

Def^n An $n \times n$ matrix is said to be lower triangular if $a_{ij} = 0$ whenever $i < j$ and it is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Finally if $a_{ij} = 0$ whenever $i \neq j$, A is called diagonal.

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e.g. $\begin{pmatrix} 5 & 2 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ is upper triangular

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal.

$a_{11} = 0 \quad a_{22} = 1$

$\begin{pmatrix} 10 & 0 & 0 \\ -2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$ is lower triangular

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is upper and lower triangular and diagonal.

For a diagonal matrix $D = (d_{ij})$ we sometimes use the notation

$D = diag(d_{11}, d_{22}, \ldots, d_{nn})$

e.g. $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} = diag(5, 3, 10)$

Def^n A is called a symmetric matrix if $A = A^T$

It is called skew-symmetric if
$A = -A^T$

e.g. $\begin{pmatrix} -7 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 0 \end{pmatrix}$ is symmetric

Say A is skew-sym. Prove that diagonal entries have to be 0.

Pf We want to show that for all possible values of i, $a_{ii} = 0$

$$a_{ii} = -a^T_{ii} = -a_{ii}$$
$$2a_{ii} = 0 \implies a_{ii} = 0$$

Def^n An $n \times 1$ matrix function is called a column n-vector function

e.g. $\begin{pmatrix} t^2 \\ \ln(3+t) \end{pmatrix}$

e.g. $\begin{pmatrix} 5 & 4 & 3 \\ 1 & 2 & 7 \end{pmatrix} + \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$

$= \begin{pmatrix} 15 & 24 & 33 \\ 41 & 52 & 67 \end{pmatrix}$

$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 10 \\ -7 \\ 3 \end{pmatrix}$ is undefined!

Properties
- $A + B = B + A$ (addition is commutative)

e.g. $10 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$

$i \begin{pmatrix} 3+i & i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3i-1 & -1 \\ 0 & i \end{pmatrix}$

Properties:
- $1A = A$ (unit property)
- $s(A+B) = sA + sB$ distributivity
- $(s+t)A = sA + tA$ //

e.g. $\begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 27 & 36 \end{pmatrix}$

$\begin{pmatrix} 5 & 7 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 7 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

called the 2×2 zero matrix

Def^n If a and b are either row or column n-vectors with components $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively, then their dot product, denoted $a \cdot b$ is the number
$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

e.g. $\begin{pmatrix} 0 & 5 & 4 \\ -5 & 0 & 6 \\ -4 & -6 & 0 \end{pmatrix}$ is skew-sym.

Note that for skew-symmetric matrices diagonal entries have to be 0.

Def^n An $m \times n$ matrix function is a rectangular array with m rows and n columns whose elements are functions of a real variable t.

e.g. $A(t) = \begin{pmatrix} 5 & e^t & \cosh(t) - 2t \\ t^2 + \frac{5}{t} & \ln t & \sin t \end{pmatrix}$

2.2 Matrix Algebra

If $A = (a_{ij})$, $B = (b_{ij})$ are both $m \times n$ matrices, we set
$$A + B = (a_{ij} + b_{ij})$$

- $A + (B + C) = (A + B) + C$
(addition is associative)

By a scalar, we mean a real or complex number.

Def^n If A is an $m \times n$ matrix and c is a scalar we set
$cA = (c\, a_{ij})$, the scalar multiplication.

- $s(tA) = (st)A = (ts)A$
$= t(sA)$
associativity

Def^n If $A = (a_{ij})$, $B = (b_{ij})$ have the same size, set
$$A - B = A + (-1)B$$
(equivalently $A - B = (a_{ij} - b_{ij})$)

We denote the $m \times n$ 0 matrix by $0_{m \times n}$ or simply by 0 if dimensions are clear.

Properties
- $A + 0 = A$
- $A - A = 0$
- $0A = 0$

scalar 0

0 matrix of size $=$ size of A

e.g. $(1\ 2\ 3)\cdot(10\ \ 0\ -10)$

$= 1(10)+2(0)+3(-10)$

$= 10+0-30 = -20$

or $\begin{pmatrix}3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\5\end{pmatrix} = -3+10 = 7$

Remark: We only take dot product of two row vectors or two column vectors.

If $x = (x_1\ x_2 \cdots x_n)$

$y = \begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$, then

their matrix product xy is the 1×1 matrix

$$xy = (x_1 \cdots x_n)\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix} = (x_1y_1 + x_2y_2 + \cdots + x_ny_n)$$