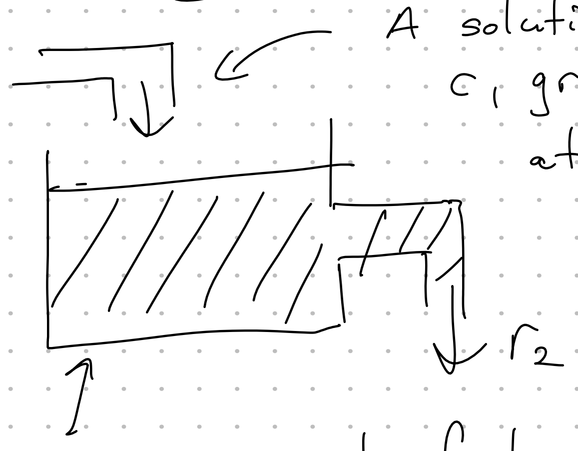
1.7 Modeling problems using 1^{st} order linear diff. eq.s

Mixing problems



A solution of concentration c_1 grams/liter flows in at a rate of r_1 liters/minute

r_2

$A(t)$: amount of chemical at time t (grams)

$V(t)$: Volume of solution at time t

$c_2(t) = \frac{A(t)}{V(t)}$: concentration of chemical inside the tank at time t.

Solution of concentration c_2 grams/liter flows out at a rate of r_2 liters/minute.

Example A tank contains 8L of water in which is dissolved 32 g of salt. A solution containing 2 g/L of salt flows into the tank at a rate of 4L/mins and the well-stirred mixture flows out at a rate of 2L/min.

1) Determine the amount of salt in the tank after 20 mins.
2) What is the concentration of salt in the tank at that time?

$V(0) = 8\text{ L}$

$V'(t) = r_{in} - r_{out} = 2\text{ L/min}$

$V = 2t + C$ Since $V(0) = 8$, $C = 8$

So $V = 8 + 2t$

$A(0) = 32\text{ g}$ $\quad A' = ?\ (\text{g/min})$

ΔA : change in the amount over a small period of time Δt

$C_{in} = 2\text{ g/L}$ $\quad r_{in} = 4\text{ L/min}$

$C_{out} = C(t)$ $\quad r_{out} = 2$

$\Delta A = (C_{in} r_{in} - C_{out} r_{out})\Delta t$

$\Delta A = (8\text{ g/min} - 2C)\Delta t$

$\frac{\Delta A}{\Delta t} = 8 - 2C$

So in the limit we have

$A' \simeq \frac{\Delta A}{\Delta t} = 8 - 2C$

$\left(C = \frac{A}{V}\right)$ $\quad \boxed{A' = 8 - 2\frac{A}{V}}$

$A' = 8 - \frac{2A}{8+2t}$ $\quad A(0) = 32$

$A' + \frac{2A}{8+2t} = 8$

(*) $A' + \left(\frac{1}{4+t}\right)A = 8$

$p(t) = \frac{1}{4+t}$ $\quad I = e^{\int p(t)dt}$

$I = e^{\int \frac{1}{t+4}dt} = e^{\ln(t+4)}$

$I = t + 4$. So (*) can be written as

$\frac{d(IA)}{dt} = 8I = 8(t+4) = 8t + 32$

$IA = \int (8t+32)\,dt = 4t^2 + 32t + C$

$A = \frac{4t^2 + 32t + C}{t+4}$

$A(0) = \frac{C}{4} = 32$ $\quad C = 128$

$A = \frac{4t^2 + 32t + 128}{t+4}$

(a) $A(20)$

$= \frac{4(400) + 32(20) + 128}{24}$

$= \frac{296}{3}\text{ g}$

(b) $C(20) = \frac{A(20)}{V(20)} = \frac{\frac{296}{3}}{48} = \frac{37}{18}\text{ g/L}$

Example A body of mass 3kg is projected vertically upward with an initial speed of 89 meters/second. The gravitational constant is $g = 9.8\text{ m/s}^2$. The magnitude of the air resistance force is equal to $k|v|$ where $k > 0$ is a constant.

1) Use Newton's 2^{nd} Law of motion to write a diff. eq. for the velocity in terms of k and v (take the positive direction to be downward)
2) Solve the diff. eq. to find a formula for the velocity at any time (in terms of k)
3) Find the limit of this velocity for a fixed time t_0 as the air resistance coefficient k goes to 0.

(1) $ma = F = mg - kv$ ← Air resistance

$a = v'$ $\quad m = 3$, $g = 9.8$

$mv' = mg - kv$

$v' = 9.8 - \frac{k}{3}v$ $\quad v(0) = -89$

(2) $v' + \frac{k}{3}v = 9.8$ $\quad p(t) = \frac{k}{3}$

$I = e^{\int p(t)dt} = e^{\frac{k}{3}t}$

$(Iv)' = 9.8I = 9.8e^{\frac{k}{3}t}$

$Iv = 9.8\left(\frac{3}{k}\right)e^{\frac{k}{3}t} + C$

$v = 9.8\left(\frac{3}{k}\right) + Ce^{-\frac{k}{3}t}$

$v(0) = 9.8\left(\frac{3}{k}\right) + C = -89$

$C = -89 - 9.8\left(\frac{3}{k}\right)$

$v = 9.8\left(\frac{3}{k}\right) + C e^{-\frac{k}{3}t}$

(3) $\lim_{k\to 0} v = \lim_{k\to 0} 9.8\left(\frac{3}{k}\right) - 89 e^{-\frac{k}{3}t} - 9.8\left(\frac{3}{k}\right) e^{-\frac{k}{3}t}$

$= -89 + \lim_{k\to 0} 9.8(3)\left(\frac{1 - e^{-\frac{k}{3}t}}{k}\right)$

$\overset{L'H}{=} -89 + \lim_{k\to 0} 9.8(3)\left(\frac{+\frac{t}{3} e^{-\frac{k}{3}t}}{1}\right)$

$= -89 + 9.8(\cancel{3})\frac{t}{\cancel{3}} = -89 + 9.8t$

Example When a hot object is placed in a water bath whose temperature is 25°C, it cools from 100°C to 50°C in 200s.

In another bath, the same cooling occurs in 180s. Find the temperature of the second bath.

$$\frac{dT}{dt} = -k(T - T_m)$$

$$\frac{1}{T - T_m}\frac{dT}{dt} = -k$$

$$\ln|T - T_m| = -kt + c$$

$$T - T_m = Ce^{-kt} \quad \text{or} \quad T = T_m + Ce^{-kt}$$

In bath #1 $T_m = 25$, $T(0) = 100$, $T(200) = 50$

$100 = 25 + Ce^0 = 25 + C$

$C = 75$

$50 = 25 + 75e^{-k(200)}$

$\frac{1}{3} = \frac{25}{75} = e^{-200k} \qquad -200k = \ln\left(\frac{1}{3}\right)$

$k = -\frac{\ln\left(\frac{1}{3}\right)^{-1}}{200} = \frac{\ln 3}{200}$

In bath #2 $T(0) = 100$ $T(180) = 50$

So $100 = T_m + Ce^0$

$100 = T_m + C$

$50 = T_m + C\underbrace{e^{-\frac{\ln 3}{200}(180)}}$

$e^{-\frac{\ln 3}{200}(180)} = e^{\left(-\frac{180}{200}\right)\ln 3} = e^{-0.9\ln 3} = e^{\ln 3^{-0.9}} = 3^{-0.9}$

$50 = T_m + 3^{-0.9} C \qquad C = 100 - T_m$

$50 = T_m + 3^{-0.9}(100 - T_m)$

$50 = T_m + 3^{-0.9}(100) - 3^{-0.9}T_m = T_m(1 - 3^{-0.9}) + 3^{-0.9}(100)$

$T_m \cong 20.382$

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ \sqrt{5} & 2 & 3 & 4 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$