Example $f(x) = \sin x$ Taylor series centered at $\pi/3$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad a = \frac{\pi}{3} \qquad c_n = \frac{f^{(n)}(a)}{n!} = \frac{f^{(n)}(\pi/3)}{n!}$$

$f(x) = \sin x$ $\qquad f(a) = \sin \pi/3 = \frac{\sqrt{3}}{2} = f^{(4)}(a)$

$f'(x) = \cos x$ $\qquad f'(a) = \cos \pi/3 = \frac{1}{2}$

$f''(x) = -\sin x$ $\qquad f''(a) = -\sin \pi/3 = -\frac{\sqrt{3}}{2}$

$f'''(x) = -\cos x$ $\qquad f'''(a) = -\cos \pi/3 = -\frac{1}{2}$

$f^{(4)}(x) = \sin x$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} \cdot \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{2} \cdot \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{\sqrt{3}}{2} \cdot \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots$$

Example Find the Maclawin series for $f(x) = (1+x)^k$ where k is any real number.

Note for $k=2$, $f(x) = (1+x)^2 = 1 + 2x + x^2 + 0x^3 + 0x^4 \ldots$

$k=3$, $f(x) = (1+x)^3 = 1 + 3x + 3x^2 + x^3 + 0x^4 + 0x^5 + \cdots$

$f(x) = (1+x)^k$ $\qquad f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$ $\qquad f'''(0) = k(k-1)(k-2)$

$\vdots$

$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$ $\qquad f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$

$$c_n = \frac{f^{(n)}(0)}{n!} \qquad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

This series is called the binomial series. If $|x| < 1$, the series converges by the ratio test. It diverges if $|x| > 1$.

Notation: $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$

"k choose n"

or binomial coefficients.

$\{a,b,c,d\}$ 4 elements choose 2

$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$

$\binom{4}{2} = 6$

Example

Find $\int e^{-x^2}\,dx$. Recall $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

Therefore $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$

$$\int e^{-x^2}\,dx = \int 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots\,dx = C + \frac{x^1}{1(0!)} - \frac{x^3}{3(1!)} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}$$

Find $\int_0^1 e^{-x^2}\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\right]_0^1 = \sum_{n=0} \frac{(-1)^n}{(2n+1)\,n!} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}$

using the first 4 terms and find an upper bound for the error.

Alternating series

So the upper bound for the error is the absolute value of the first term we omitted.

$$|\text{Error}| \leq \frac{1}{9(4!)} = \frac{1}{9(24)}$$

Example Evaluate

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{\left(\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \cancel{-1} \cancel{-x}}{x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right)}{\cancel{x^2}} = \frac{1}{2}$$

Example Find the first three nonzero term in the Maclawin series for $e^x \sin x$.

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= x + x^2 + \frac{x^3}{2!} - \frac{x^3}{3!} + \cdots = x + x^2 + x^3\left(\frac{1}{2} - \frac{1}{6}\right) + \cdots$$

$$= \underbrace{x + x^2 + \frac{1}{3}x^3}_{} + \cdots$$

11.11 Applications of Taylor Polynomials (Not included in the final)

Example a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

b) How accurate is this approximation when $7 \leq x \leq 9$?

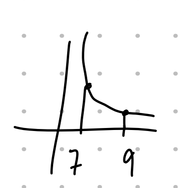
(a) $f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad f(a) = \sqrt[3]{8} = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(a) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}2^{-2} = \frac{1}{12}$

$f''(x) = \frac{-2}{9}x^{-5/3}$ $\qquad f''(a) = -\frac{2}{9}(8^{-5/3}) = \frac{-2}{9}(2)^{-5} = -\frac{1}{144}$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(b) $f'''(x) = \frac{10}{27}x^{-8/3}$ $\qquad |f'''(x)| \leq f'''(7) = \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021 = M$



Remainder

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3 \leq \frac{0.0021}{6} \cdot 1^3 = \frac{0.0021}{6}. \qquad 7 \leq x \leq 9$$