Example a) Evaluate $\int \frac{1}{1+x^7}dx$ as a power series

b) Use "the first three terms" in part (a) to approximate $\int_0^{0.5} \frac{1}{1+x^7}dx$. What is the error?

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = 1 - x^7 + x^{14} - x^{21} + \cdots = \sum_{n=0}^{\infty}(-x^7)^n$$

$$((-x^7)^n = (-1)^n x^{7n}$$

So $\int \frac{1}{1+x^7}dx = \int (1 - x^7 + x^{14} - x^{21} + \cdots)dx = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$

$$= C + \sum_{n=0}^{\infty} \int (-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$\int_0^{0.5} \frac{1}{1+x^7}dx = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \Big|_0^{0.5}$$

$$= \cancel{C} + 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} - \frac{(0.5)^{22}}{22} + \cdots - \cancel{C}$$

$$\simeq 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15}$$

alternating series the error is the absolute value of the first omitted term.

So $|\text{Error}| \le \frac{(0.5)^{22}}{22}$

11.10 Taylor and Maclaurin Series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots$$

$$f''(a) = 2c_2$$

$$f'''(a) = 6c_3 = 3\cdot 2\, c_3 \quad \cdots \quad f^{(n)}(a) = n!\,c_n$$

So $c_n = \frac{f^{(n)}(a)}{n!}$ $\qquad 0! = 1$

Theorem If f has a power series expansion at a, that is, if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ $\quad |x-a| < R$

then its coeff. are given by $c_n = \frac{f^{(n)}(a)}{n!}$

Then it follows that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This is called the Taylor series of f at a.

If $a=0$, it is called a Maclaurin series.

Example Find the Maclaurin series of $f(x)=e^x$ and its radius of convergence.

$f(a) = f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

In general, $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = 1$. Thus, $c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

$$e^x = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$a_n = \frac{x^n}{n!} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ for any value of x, the series is convergent for all x. In other words, $R = \infty$.

Q) $f(x) \overset{?}{=} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \underbrace{f(a) + f'(a)(x-a)}_{\text{linear approximation}} + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \cdots$$

1st order

In general, we may define

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the n^{th} degree Taylor polynomial of f at a.

In general, $f(x)$ is the sum of its Taylor series if

$$\boxed{f(x) = \lim_{n\to\infty} T_n(x)}$$

let $R_n(x) = f(x) - T_n(x)$ so $f(x) = T_n(x) + R_n(x)$

$R_n(x)$ is called the remainder of the Taylor series.

Note that if $\lim_{n\to\infty} R_n(x) = 0$ then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}\left(f(x)-R_n(x)\right) = f(x) - \lim_{n\to\infty} R_n(x) = f(x).$$

Theorem If $f(x) = T_n(x) + R_n(x)$ and

$\lim_{n\to\infty} R_n(x) = 0$ for $|x-a|<R$

then f is equal to its Taylor series for $|x-a|<R$.

Theorem (Taylor's Inequality) If $|f^{(n+1)}(x)| \le M$ for $|x-a|\le d$, then the Remainder $R_n(x)$ satisfies

$a-d \quad a \quad a+d \qquad |R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a|\le d$.

e.g. for $n=1$, say $|f''(x)| \le M$ for $a \le x \le a+d$

then in particular $f''(x) \le M$

So

$$\int_a^x f''(t)\,dx \le \int_a^x M\,dt$$

$$f'(x) - f'(a) \le M(x-a) \quad \text{or} \quad f'(x) \le f'(a) + M(x-a)$$

$$\int_a^x f'(t)\,dt \le \int_a^x \left(f'(a) + M(t-a)\right)dt$$

$$f(x) - f(a) \le f'(a)(x-a) + \frac{M(x-a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x-a) \le \frac{M}{2}(x-a)^2$$

$$\underbrace{f(x) - \underbrace{\left(f(a) + f'(a)(x-a)\right)}_{T_1(x)}}_{R_1(x)} \le \frac{M}{2}(x-a)^2$$

Example Prove that e^x is equal to the sum of its Maclaurin series

Recall $f^{(n)}(x) = e^x \quad f^{(n)}(0) = 1 \qquad (f(x)=e^x)$

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$R_n(x) = f(x) - T_n(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)$$

Recall for $a=0$: If $|f^{(n+1)}(x)| \le M$ for $|x|\le d$, ← $[-d, d]$

then the Remainder $R_n(x)$ satisfies

$|R_n(x)| \le \dfrac{M}{(n+1)!}|x|^{n+1}$ for $|x|\le d$.

$f^{(n+1)}(x) = e^x$ so if $|x|\le d$ $\quad |f^{(n+1)}(x)| = |e^x| = |e^d| = e^d$

So $|R_n(x)| \le \dfrac{e^d|x|^{n+1}}{(n+1)!} \rightsquigarrow -\dfrac{e^d|x|^{n+1}}{(n+1)!} \le R_n(x) \le \dfrac{e^d|x|^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{e^d|x|^{n+1}}{(n+1)!} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^d \lim_{n\to\infty} \frac{|x|}{1}\frac{|x|}{2}\cdots\frac{|x|}{(n+1)} \quad <1$$

$$= 0$$

So by the Squeeze thm, $\lim_{n\to\infty} R_n(x) = 0$ for $|x|\le d$

Since d is arbitrary, this shows that e^x is equal to its Maclaurin series for all x. $(-\infty, \infty)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

e.g. $\quad e^1 = e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \cdots$