$e^x = \sum_{n=0}^{\infty} c_n x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ Maclaurian $\Leftrightarrow$ $a=0$

$c_n = \frac{1}{n!}$

Example Find the Maclaurin series for $\sin x$ and prove that it is equal to $\sin x$ for all x.

$a=0$

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad c_n = \frac{f^{(n)}(a)}{n!}$

$$\left.\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \end{array}\right]$$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

Since $f^{(4)}(x) = \sin x = f(x)$ $\leftarrow$ these values will keep repeating.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + \frac{1}{1!}x^1 + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To show $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ we need to use Taylor's inequality.

$f^{(n+1)}(x)$ is either $\pm\sin x$ or $\pm\cos x$. In any case, $|f^{(n+1)}(x)| \le 1 = M$ for all x.

$$|R_n(x)| \le \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x^{n+1}|}{(n+1)!} \longrightarrow 0$$ by Ratio test.

So $|R_n(x)| \to 0$ for all x. Thus, $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Example Find the Maclaurin series for $\cos x$.

We can follow the above procedure or simply note that

$$\cos x = (\sin x)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)' = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$ with the same radius of convergence ($R = \infty$)

Example Find the Maclaurin series for the function $f(x) = x\cos x$.

$$f(x) = x\cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \qquad (R = \infty)$$