$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Example For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

$$a_n = \frac{(x-3)^n}{n} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right| = \left|\frac{(x-3)^{\cancel{n+1}}}{\cancel{(x-3)^n}} \cdot \frac{n}{n+1}\right|$$

$$= |x-3| \frac{n}{n+1} \qquad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x-3| \frac{n}{n+1}$$

$$= |x-3| \lim_{n\to\infty} \frac{n}{n+1} = |x-3|$$

If $|x-3| < 1$ then the series is ABS. CONV.

If $|x-3| > 1$ " " DIV.

If $|x-3| = 1$ the ratio test is inconclusive

However, in this case we have either $x-3=1$ or $x-3=-1$

If $x-3=1$ (or $x=4$) then $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ ← the harmonic series it is DIV

If $x-3=-1$ (or $x=2$) then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← the alternating harmonic series it is CONV.

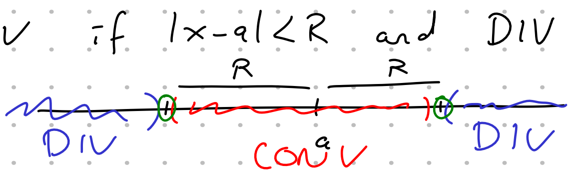
So if we set $f(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ then its domain is $[2,4)$

Theorem For $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are 3 possibilities:

a) The series conv only when $x=a$. $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + \cancel{c_1(x-a)} + \cancel{c_2(x-a)^2 + \cdots}$

b) The series conv for all x

c) There is a positive number R such that the series CONV if $|x-a|<R$ and DIV if $|x-a|>R$



At the boundary anything can happen.

R is called the radius of convergence.

for (a) we set $R=0$, for (b) we set $R=\infty$

Example Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $\qquad a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n x^n}{\sqrt{n+1}}}\right| = \left|\frac{(-3)^{\cancel{n+1}}}{\cancel{(-3)^n}} \cdot \frac{x^{\cancel{n+1}}}{\cancel{x^n}} \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}}\right|$$

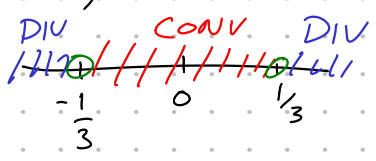
$$= |-3x| \sqrt{\frac{n+1}{n+2}} \qquad \lim_{n\to\infty} |+3x| \sqrt{\frac{n+1}{n+2}} = |+3x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}$$

$$= |3x| \sqrt{\underbrace{\lim_{x\to\infty} \frac{n+1}{n+2}}_{=1}} = |3x|$$

So if $|3x|<1$, the series converges at x.

$$|x| < \frac{1}{3} = R \qquad (a=0 \quad |x-a|<R)$$

The radius of convergence is $\frac{1}{3}$.



Now we check $x=\frac{1}{3}$ and $x=-\frac{1}{3}$.

If $x=\frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

which is CONV by the alternating series test.

If $x=-\frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n(-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(+1)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$

which is DIV by the limit comparison test $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$

So the Interval of Convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

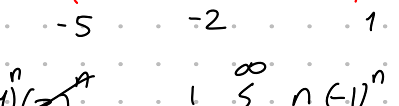
Example $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ $\qquad a_n = \frac{n(x+2)^n}{3^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)}{n} \frac{(x+2)^{\cancel{n+1}}}{\cancel{(x+2)^n}} \cdot \frac{\cancel{3^{n+1}}}{3^{\cancel{n+2}}}\right| = \frac{n+1}{n} \cdot \frac{1}{3} |x+2|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+2|}{3} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+2|}{3}$$

$\frac{|x+2|}{3} < 1$ so $|x+2| < 3$ so the radius of convergence $R=3$.

$|x-a|<R$, $a=-2$



when $x=-5$, we have

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-1)^n \cancel{(3)^n}}{3^{\cancel{n+1}}} = \frac{1}{3} \sum_{n=0}^{\infty} n(-1)^n$$

Since $\lim_{n\to\infty} n(-1)^n$ Does not exist, the series is DIV.

when $x=1$, we have $\sum_{n=0}^{\infty} \frac{n\cancel{(3)^n}}{3^{\cancel{n+1}}} = \frac{1}{3} \sum_{n=0}^{\infty} n$ DIV

So the interval of conv is $(-5, 1)$.

11.9 Representations of Functions as Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

Example Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 \cdots$$

$$= \sum_{n=0}^{\infty} (-x^2)^n \quad \text{is convergent for } |-x^2|<1$$

So $|x|<1$ and the radius of convergence is 1.

If $x=1 \Rightarrow -x^2 = -1$ so $\sum (-x^2)^n = \sum (-1)^n$ DIV

If $x=-1 \Rightarrow -x^2 = -1$ so $\sum (-x^2)^n = \sum (-1)^n$ DIV

Example $\frac{1}{x+2}$. Express the function as a power series.

$$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2}\cdot\frac{1}{(1+\frac{x}{2})} = \frac{1}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right)$$

$$= \frac{1}{2}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right] = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

Example

$$\frac{x^3}{x+2} = x^3\left(\frac{1}{x+2}\right) = x^3\,\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n x^{n+3}$$

$$\boxed{\left(\frac{-x}{2}\right)^n = \left(\frac{-1}{2}\right)^n x^n}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Theorem If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and continuous) on the interval $(a-R, a+R)$ and

a) $f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

b) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of conv. for (a) and (b) are both R.