Example

Express $\frac{1}{(1-x)^2}$ as a power series.

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$

$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = \frac{+1}{(1-x)^2}$

$= \left(1 + x + x^2 + x^3 + \cdots\right)' = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$

$= \sum_{n=0}^{\infty} (n+1) x^n \qquad |x| < 1 \quad$ So $R = 1$.

Example Find a power series expansion for $\ln(1+x)$. $\quad |x| < 1$

$(\ln(1+x))' = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} (-x)^n$

So $\ln(1+x) = \int \frac{1}{1+x} dx + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$

$= C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad |x| < 1 = R$

Since at $x = 0$ $\ln(1+x) = \ln 1 = 0 = C = C + x - \frac{x^2}{2} + \cdots$

Example Same question for $\tan^{-1}(x)$. $\quad |x| < 1$

$(\tan^{-1}(x))' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$

$\tan^{-1}(x) = C + \int 1 - x^2 + x^4 \cdots \, dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$|x| < 1 = R$

Since $\tan^{-1}(0) = 0$, $\quad 0 = C + 0 - \frac{0^3}{3} + \frac{0^5}{5} \cdots$

so $C = 0$

e.g. $\frac{1}{1-(2x)} = 1 + 2x + 4x^2 + 8x^3 + \cdots \qquad |2x| < 1$

$|x| < \frac{1}{2} = R$