The Root Test

$\sum a_n$ $\qquad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \begin{cases} L<1 & \text{then } \sum a_n \text{ is ABS. CONV.}\\ L>1 \text{ (or } \infty) & \text{then } \sum a_n \text{ is DIV}\\ L=1 & \text{then the test is inconclusive}\end{cases}$

Example $\sum_{n=0}^{\infty} ar^n \quad (a,r>0)$ $\qquad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}(|ar^n|)^{1/n}$

$= \lim_{n\to\infty}(ar^n)^{1/n} = \lim_{n\to\infty} a^{1/n}(r^n)^{1/n} = \lim_{n\to\infty} a^{1/n} r$

$= a^{\left(\lim_{n\to\infty}\frac{1}{n}\right)} r = a^0 r = r$ $\quad$ if $r<1$ then $\sum ar^n$ is CONV

if $r>1$ then $\sum ar^n$ is DIV

if $r=1$ (or -1) then the root test is inconclusive

(however we know that a geometric series with $|r|=1$ is DIV.)

Example $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ $\qquad a_n = \left(\frac{2n+3}{3n+2}\right)^n > 0$

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\frac{2n+3}{3n+2}\right)^{\cancel{n}\cdot\frac{1}{\cancel{n}}} = \frac{2}{3} < 1$

So $\sum_{n=1} a_n$ is ABS. CONV. (and therefore CONV.)

Example $\sum_{n=1}^{\infty}\underbrace{\frac{n-1}{2n+1}}_{a_n}$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\frac{1}{n}(n-1)}{\frac{1}{n}(2n+1)}$

$= \lim_{n\to\infty}\frac{1-\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2}$

So $\sum_{n=1} a_n$ is DIV

Example $\sum_{n=1}^{\infty}\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ $\qquad a_n = \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \sim \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3}$

$= \frac{1}{3n^{3/2}} = b_n$ $\qquad \sum_{n=1}^{\infty} b_n = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ $\qquad p=\frac{3}{2}>1$ so $\sum b_n$ is CONV

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{3n^{3/2}}} = \lim_{n\to\infty}\frac{3n^{3/2}(n^3+1)^{1/2}}{3n^3+4n^2+2}$

$= \lim_{n\to\infty}\frac{\frac{1}{n^3}\cancel{3}n^{3/2}(n^3+1)^{1/2}}{\frac{1}{n^3}\cancel{3}(n^3+\frac{4}{3}n^2+\frac{2}{3})} = \lim_{n\to\infty}\frac{n^{-3/2}(n^3+1)^{1/2}}{1+\frac{4}{3n}+\frac{2}{3n^3}}$

$= \lim_{n\to\infty}\frac{(n^{-3}(n^3+1))^{1/2}}{1+\frac{4}{3n}+\frac{2}{3n^3}} = \lim_{n\to\infty}\frac{(1+\frac{1}{n^3})^{1/2}}{1+0+0} = \frac{1}{1} = 1$

Thus, by the limit comparison, $\sum a_n$ is also CONV.

Example $\sum_{n=1}^{\infty} ne^{-n^2}$ $\qquad f(x) = xe^{-x^2}$ $\qquad a_n = ne^{-n^2} = f(n)$

Method 1: Integral test

$\int_1^{\infty} xe^{-x^2}\,dx$ $\qquad u=-x^2$, $du = -2x\,dx$

$-\frac{1}{2}\int_{-1}^{-\infty} e^u\,du = \frac{1}{2}\int_{-\infty}^{-1} e^u\,du = \frac{1}{2}e^u\Big|_{-\infty}^{-1}$

$= \frac{1}{2}e^{-1} - \lim_{t\to-\infty}\frac{1}{2}e^t = \frac{1}{2}e^{-1}$ $\qquad$ So $\int_1^{\infty} f(x)\,dx$ is CONV

thus, $\sum_{n=1}^{\infty} a_n$ is CONV by the integral test.

Method 2 Ratio test

$\sum_{n=1}^{\infty} ne^{-n} = \sum_{n=1}^{\infty}\frac{n}{e^n}$ $\qquad a_n = \frac{n}{e^n}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{e^n}{e^{n+1}} = \frac{1}{e} < 1$

Thus, $\sum_{n=1}^{\infty} ne^{-n}$ ABS. CONV.

Example $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$ $\qquad \left|(-1)^n\frac{n^3}{n^4+1}\right| = \frac{n^3}{n^4+1} \sim \frac{n^3}{n^4} = \frac{1}{n}$

$\sum\frac{1}{n}$ is DIV

This is an alternating series.

$\frac{n^3}{n^4+1} = f(n)$, $f(x) = \frac{x^3}{x^4+1}$ is cont. and decreasing. $\lim_{n\to\infty}\frac{n^3}{n^4+1} = 0$

$f'(x) = \frac{3x^2(x^4+1) - x^3(4x^3)}{(x^4+1)^2} = \frac{3x^6+3x^2-4x^6}{(x^4+1)^2} = \frac{-x^6+3x^2}{(x^4+1)^2}$

$\frac{-x^6+3x^2}{(x^4+1)^2} = \frac{x^2(3-x^4)}{(x^4+1)^2} < 0$ if $3-x^4<0$

if $x^4>3$

if $x>3^{1/4}$

So by the alt. series test $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$ is CONV.

Example $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ $\qquad a_n = \frac{2^n}{n!}$ $\qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$

$= \lim_{n\to\infty}\frac{\cancel{2^{n+1}}}{\cancel{2^n}}\cdot\frac{\cancel{n!}}{\cancel{(n+1)!}} = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$

So $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ is CONV.

Power Series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = f(x)$$

Domain of f is those x values at which the series is conv.

e.g. If $c_n = 1$ forall n, then

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$ is a geometric series with common ratio $= x$

So it is CONV if and only if $|x|<1$. In other words,

the domain of $f(x) = 1 + x + x^2 + \cdots$ is $|x|<1$ $\quad (-1,1)$

In fact, for $x \underset{\text{in}}{\in} (-1,1)$ $\quad f(x) = \frac{1}{1-x}$ $\qquad \frac{a=1}{1-r=x}$

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at a.

Example For what values of x, $\sum_{n=0}^{\infty} n!\, x^n$ CONV?

$a_n = n!\, x^n \qquad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right|$

$= \lim_{n\to\infty} |(n+1)\, x| = \lim_{n\to\infty} (n+1)|x| = \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{if } x\neq 0 \end{cases}$

So only for $x=0$ $\sum a_n$ is CONV. For $x \neq 0$, $\sum a_n$ is DIV.