Example $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ is this CONV or DIV?

let $a_n = (-1)^n \dfrac{\pi^{2n}}{(2n)!}$ $\qquad \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{\pi^{2(n+1)}}{(2(n+1))!}}{\frac{\pi^{2n}}{(2n)!}}$

$$= \lim_{n\to\infty} \frac{\pi^{\cancel{2n+2}}}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{\pi^{2n}}} = \lim_{n\to\infty} \frac{\pi^2 \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = 0 < 1$$

So by the ratio test, $\sum a_n$ is ABS. CONV (and therefore CONV.)

Example $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ $\qquad a_n = \dfrac{n^{2n}}{(1+n)^{3n}}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n^{2n}}{(1+n)^{3n}}\right)^{1/n} = \lim_{n\to\infty} \frac{(n^{2n})^{1/n}}{((1+n)^{3n})^{1/n}}$$

$$= \lim_{n\to\infty} \frac{n^2}{(1+n)^3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\left(\frac{1+n}{n}\right)^3} = \lim_{n\to\infty} \frac{\overset{0}{\frac{1}{n}}}{(\underset{0}{\frac{1}{n}}+1)^3}$$

$$= \frac{0}{(0+1)^3} = 0 < 1$$

So $\sum a_n$ is ABS. CONV. (and CONV) by the root test.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$

$$0 \le b_n = \left|\frac{\sin(2n)}{1+2^n}\right| = \frac{|\sin 2n|}{1+2^n} \le \frac{1}{1+2^n} < \frac{1}{2^n}$$

$\sum \frac{1}{2^n}$ is a geom. series with $r = \frac{1}{2}$ so $|r| < 1$ and it is CONV. So by the direct comparison $\sum \left|\frac{\sin 2n}{1+2^n}\right|$ is CONV.

So $\sum \frac{\sin 2n}{1+2^n}$ is ABS. CONV. (and therefore CONV.)

Example $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n \cos\left(\frac{1}{n^2}\right)}_{\text{DIV}}$ $\qquad \displaystyle\lim_{n\to\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$ $\quad \cos(0) = 1$

$\sum a_n$ is DIV by the test for DIV.

$\sum a_n$ is ABS CONV if $\sum |a_n|$ is CONV

$\Downarrow$ ~~$\sum a_n$ is CONV~~

e.g. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right) \to 1$ $\qquad \left|\cos\left(\frac{1}{n^2}\right)\right| = \cos\left(\frac{1}{n^2}\right)$ if n is large enough $\qquad \cos\left(\frac{1}{n^2}\right) \to 1$

$\sum \cos\left(\frac{1}{n^2}\right)$ is again DIV

After class discussion

$$\boxed{\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,dx}$$