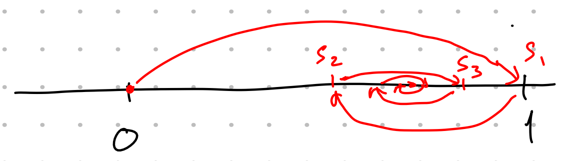
11.5 Alternating Series

$2 - 3 + \frac{1}{2} - \frac{1}{3} + \sqrt{2} - \pi \ldots - -$

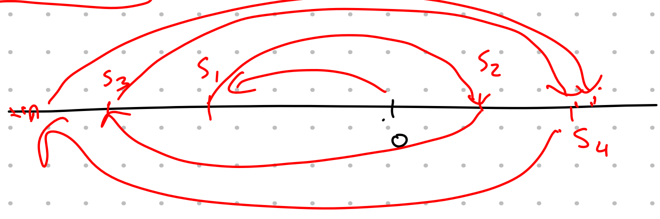
$+ \quad - \quad + \quad - \quad - \quad - \quad -$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$s_1 = 1$
$s_2 = 1 - \frac{1}{2}$
$s_3 = 1 - \frac{1}{2} + \frac{1}{3}$



$$\underbrace{-\frac{1}{2} + \frac{2}{3}} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} \left((-1)^n \frac{n}{n+1} \right) = a_n$$



$\lim_{n\to\infty} (-1)^n \frac{n}{n+1}$ Does Not Exist

$\sum_{n=1}^{\infty} a_n$ is DIV

Alternating Series Test

If the alternating series satisfies $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ (where $b_n > 0$ for all n)

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is CONV.

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ "alternating harmonic series"

$(-1)^{n-1} \frac{1}{n} = (-1)^{n-1} b_n \qquad b_n = \frac{1}{n}$

(i) $b_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = b_n$ ✓

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

So the alternating harmonic series is CONV.

Example $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating.

$b_n = \frac{3n}{4n-1} \qquad \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$

So the series is DIV by the Test for Divergence.

Example $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is it DIV or CONV?

The series is alternating with $b_n = \frac{n^2}{n^3+1}$

We need to check

(i) $b_{n+1} \le b_n$? $\qquad$ (ii) $\lim_{n\to\infty} b_n = 0$?

$f(x) = \frac{x^2}{x^3+1} \qquad b_n = f(n) \qquad \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\overset{\nearrow 0}{\frac{1}{n}}}{1 + \underset{\searrow 0}{\frac{1}{n^3}}} = 0$ ✓

If $f'(x) \le 0$ then f decreasing

so $f(n+1) = b_{n+1} \le f(n) = b_n$ so let's show that

$f'(x) \le 0$ $\boxed{\text{for } x \ge 1}$ $\qquad f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2}$

$$f'(x) = \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} = \frac{-x^4 + 2x}{(x^3+1)^2} = \frac{x(-x^3+2)}{(x^3+1)^2}$$

so $f'(x) \le 0$ if and only if $(-x^3+2) \le 0$ if and only if $2 \le x^3$ $\qquad \sqrt[3]{2} \le x$ ✓

Therefore the series is CONV.

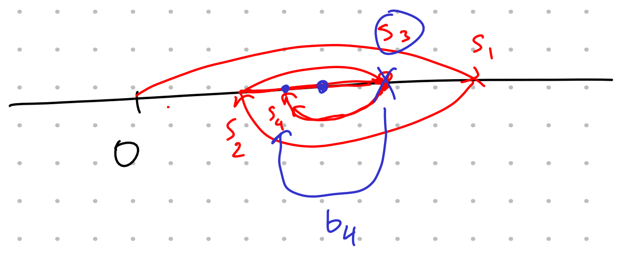
Estimating Sums

Alternating Series Estimation Theorem

If $s = \sum (-1)^{n-1} b_n$ where $b_n > 0$ and

(i) $b_{n+1} \le b_n$ $\qquad$ (ii) $\lim_{n\to\infty} b_n = 0$

then $\qquad |R_n| = |s - s_n| \le b_{n+1}$



Example Estimate the error for using the first three terms to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \underbrace{\frac{-1}{1!} + \frac{1}{2!} - \frac{1}{3!}}_{s_3} + \boxed{\frac{1}{4!}} - \cdots$

$$|R_3| \le |a_4| = \left| \frac{(-1)^4}{4!} \right| = \frac{1}{4!} = \underline{\underline{\frac{1}{24}}}$$

11.6 Absolute Convergence (+ −)

Given $\sum_{n=1}^{\infty} a_n$, we can consider the following series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

Def^n $\sum_{n=1}^{\infty} a_n$ is called <u>absolutely convergent</u> if the series of absolute value $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

is <u>absolutely convergent</u> since $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ ← CONV. $p = 2 > 1$

Example $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is CONV.

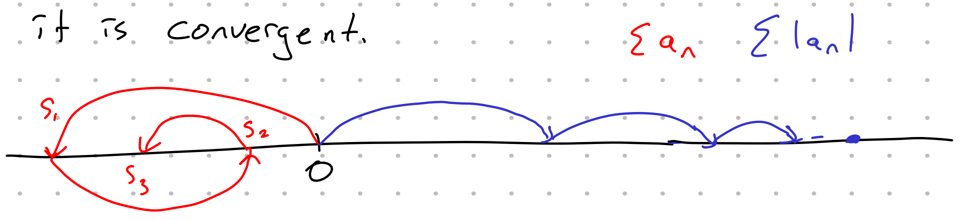
However, $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ ← "the harmonic series" DIV.

Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely conv.

Such series are called <u>conditionally convergent.</u>

More precisely, If $\sum a_n$ is <u>conv</u> but <u>not absolutely conv</u> then $\sum a_n$ is called <u>conditionally convergent</u>.

Theorem If $\sum a_n$ is absolutely convergent, then it is convergent.



Example $\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$

Determine if this series is CONV or DIV.

$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ $\qquad 0 \le |\cos n| \le 1$

$\left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$ since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is CONV

$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is CONV by the <u>direct comparison test.</u>

So $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is ABS. CONV and therefore CONV by the above theorem

Remark: The direct comparison test can only be applied for series with positive terms.