The Ratio Test

$\sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \text{Exists and } < 1 \\ (\text{Exists and } > 1) \text{ or DNE} \\ \text{Exists and } = 1 \end{cases}$$

then

- $\sum a_n$ is ABS. CONV.
- $\sum a_n$ is DIV
- Test is inconclusive

Example

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n} \qquad a_n = (-1)^n \frac{n^3}{3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{3^n}{3^{n+1}} \cdot \frac{(n+1)^3}{n^3} = \frac{1\cancel{3^n}}{\cancel{3^n}\cdot 3} \cdot \left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}\left(\lim_{n\to\infty} 1 + \underbrace{\frac{1}{n}}_{0}\right)^3 = \frac{1}{3} < 1$$

So $\sum a_n$ is ABS. CONV. In particular it is CONV.

Example

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \qquad a_n = \frac{n^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} = \frac{(n+1)^n \cancel{(n+1)}}{n^n} \cdot \frac{\cancel{n!}}{\cancel{(n+1)\cdot n!}}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \overset{\approx 2.71}{e} > 1$$

So $\sum a_n$ is DIV by the ratio test.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)}$$

$$= e^{\lim_{n\to\infty}\left(n\ln\left(1+\frac{1}{n}\right)\right)} = e^{\lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)} \overset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(\frac{1}{x}\right)'}}{\cancel{\left(\frac{1}{x}\right)'}}}$$

$$= e^{\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}} = e^1 = e$$

WW #10 Problem 2)

d)

$$\sum_{n=5}^{\infty} \frac{9^n}{10^n} = \sum_{n=5}^{\infty}\left(\frac{9}{10}\right)^n = \left(\frac{9}{10}\right)^5 + \left(\frac{9}{10}\right)^6 + \left(\frac{9}{10}\right)^7 + \dots$$

← geom. series

$$a + ar + ar^2 + \dots$$

$$a = \left(\frac{9}{10}\right)^5 \qquad ar = \left(\frac{9}{10}\right)^6$$

$$r = \frac{9}{10} \qquad \frac{a}{1-r} = \frac{\left(\frac{9}{10}\right)^5}{1-\frac{9}{10}}$$