Estimating the Sum of a Series

Say $\sum_{n=1}^{\infty} a_n$ is CONV $\quad \sum_{n=1}^{\infty} a_n = s$ ← the sum

If we estimate s by s_n (the n^{th} partial sum) what is the error? Define the remainder $R_n = s - s_n$
$= a_{n+1} + a_{n+2} + \cdots$

Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, f is cont., positive, decreasing for $x \geq n$ and $\sum_{k=1}^{\infty} a_k$ is CONV. Then

(*) lower bound → $$\underbrace{\int_{n+1}^{\infty} f(x)\,dx}_{\text{at least}} \leq R_n \leq \underbrace{\int_{n}^{\infty} f(x)\,dx}_{\text{at most}}$$ ← upper bound

Example a) Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the first 10 terms.

b) Estimate the error.

c) How many terms are required to ensure that the sum is accurate to within 0.0005?

ⓐ $$\sum_{n=1}^{10} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \cong 1.1975$$ ← calculator

ⓑ $$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \int_{10}^{\infty} x^{-3}\,dx = -\frac{x^{-2}}{2}\Big|_{10}^{\infty} = \lim_{t\to\infty}\Big(-\underbrace{\frac{t^{-2}}{2}}_{0} + \frac{10^{-2}}{2}\Big)$$
$$= \frac{1}{2}\cdot 10^{-2} = \frac{1}{200} = 0.005$$

ⓒ $$R_n \leq \int_{n}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2}n^{-2} = \frac{1}{2n^2} < 0.0005$$ ← we impose this.

$$2n^2 > \frac{1}{0.0005} = 2000 \quad \Rightarrow \quad n^2 > 1000$$
$$n > \sqrt{1000} \cong 31.6$$
n is at least 32.

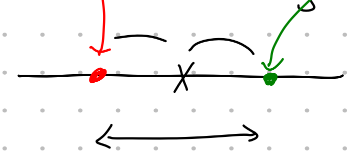
Now add s_n to all 3 sides of (*)

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq \underbrace{s_n + R_n}_{=\,s} \leq s_n + \int_{n}^{\infty} f(x)\,dx$$

for $n = 10$

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3}\,dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\,dx$$
$$1.201664 \leq s \leq 1.202532$$

So s is in a range of $(\quad) - (\quad) < 0.001$



< 0.001

If we estimate s as the average of these two numbers then the error is $< \frac{0.001}{2} = 0.0005$

The Comparison Tests

The (Direct) Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is CONV and $a_n \leq b_n$ for all n, then $\sum a_n$ is CONV

(ii) If $\sum a_n$ is DIV and $a_n \leq b_n$ for all n, then $\sum b_n$ is DIV

Example Is $\sum_{n=1}^{\infty} \underbrace{\frac{5}{2n^2+4n+3}}_{=a_n}$ CONV or DIV?

$$2n^2 \geq 2n^2 \Rightarrow 2n^2 + 4n + 3 \geq 2n^2$$

So $$0 \leq \frac{1}{2n^2+4n+3} \leq \frac{1}{2n^2} \qquad \text{So } 0 \leq \frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2} = \underbrace{\frac{5}{2}\cdot\frac{1}{n^2}}_{=b_n}$$

Since $\sum_{n=1}^{\infty} \overbrace{\frac{5}{2}\cdot\frac{1}{n^2}}^{b_n} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ is CONV,

$\sum_{n=1}^{\infty} \underbrace{\frac{5}{2n^2+4n+3}}_{a_n}$ is also CONV.

Recall $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is CONV if and only if $p > 1$

Example $\sum_{k=1}^{\infty} \frac{\ln k}{k}$.

$$\frac{\ln k}{k} \geq \frac{1}{k} \quad \text{if } k \geq 3$$

$\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{n=1}^{\infty} \frac{1}{n}$ ← $p = 1$ so this is DIV.

So $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is also DIV.

Example $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ $\qquad 2^n + 1 \geq 2^n$

$$\frac{1}{2^n+1} \leq \frac{1}{2^n}$$

So here $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $|r=\frac{1}{2}|<1$

So it is CONV.. Thus, $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is also CONV.

What about $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ in this case we need a more powerful theorem to get an easy answer.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ is finite number which is positive.

$\cancel{0}$ $\cancel{\infty}$

Then either both series CONV or both DIV.

Example $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ $\quad a_n = \frac{1}{2^n-1} \sim \frac{1}{2^n} = b_n$

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$

$$= \lim_{n\to\infty} \frac{\frac{2^n}{2^n}}{\frac{2^n}{2^n}-\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{1-\underbrace{\frac{1}{2^n}}_{\to 0}} = \frac{1}{1} = 1 \neq 0$$

So since $\sum \frac{1}{2^n}$ is CONV $\sum \frac{1}{2^n-1}$ is also CONV.

Example Is $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is CONV or DIV?

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}} \sim b_n = \frac{2n^2}{\sqrt{n^5}} = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

"highest order terms."

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{2}{n^{1/2}}} = \lim_{n\to\infty} \left(\frac{n^{1/2}}{2}\right) \frac{(2n^2+3n)}{\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{n^{5/2}+\frac{3}{2}n^{3/2}}{\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \frac{(n^{5/2}+\frac{3}{2}n^{3/2})}{\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{3}{2n}}{\sqrt{\frac{1}{n^5}(5+n^5)}} = \lim_{n\to\infty} \frac{1+\underbrace{\frac{3}{2n}}_{\to 0}}{\sqrt{\underbrace{\frac{5}{n^5}}_{\to 0}+1}} = 1 \neq 0$$

$\sum b_n = \sum \frac{2}{n^{1/2}}$ is DIV since $p = 1/2 < 1$

Thus, the original series is also DIV.

Estimating Sums

Say $\sum a_n$ and $\sum b_n$ are CONV, $0 \le a_n \le b_n$ for all n. Then if

$\sum a_n = s$ $\qquad R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$

$\sum b_n = t$ $\qquad T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$

Then $R_n \le T_n$

Example Use the first 100 terms to approximate the sum of $\sum \frac{1}{n^3+1}$. Estimate the error involved in this approximation.

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

$$R_n \le T_n \le \int_n^{\infty} \frac{1}{x^3}\,dx$$

$n=100$ $\qquad R_{100} \le T_{100} \le \int_{100}^{\infty} \frac{1}{x^3}\,dx = -\frac{x^{-2}}{2}\Big|_{100}^{\infty}$

$$= \frac{1}{2(100)^2}$$