Thm If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

$-|a_n| \le a_n \le |a_n|$ by the Squeeze thm $a_n \to 0$ as $n \to \infty$.

$\downarrow 0 \qquad \downarrow 0$

Example Find $\lim_{n\to\infty} \frac{(-1)^n}{n}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{So} \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Theorem If $\lim_{n\to\infty} a_n = L$ and the function is continuous at L then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

Example Find $\lim_{n\to\infty} \cos\left(\frac{\pi+1}{n}\right) = \cos\Big(\underbrace{\lim_{n\to\infty} \frac{\pi+1}{n}}_{=0}\Big)$

$= \cos 0 = 1$

Example Consider $a_n = \frac{n!}{n^n}$ $\qquad n! = 1\cdot 2\cdot 3 \cdot \cdots \cdot (n-1)\, n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{1}{n}\right)\cdot\left(\frac{2}{n}\right)\cdot\frac{3}{n}\cdot \cdots \cdot \frac{(n-1)}{n}\cdot\left(\frac{n}{n}\right)$$

$$0 < a_n = \frac{1}{n}\Big(\underbrace{\frac{2\cdot 3 \cdot \cdots \cdot n}{n\ n\ \cdots\ n}}_{\le 1}\Big) < \frac{1}{n}$$

$0 < a_n < \frac{1}{n}$

$\downarrow 0 \qquad \downarrow 0$

0 as well.

Example $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$$

If $r > 1$ then $r^n \to \infty$ DIV

If $r = 1$ then $r^n = 1^n = 1 \to 1$ CONV

If $-1<r<1$ then $r^n \to 0$ CONV

If $r=-1$ then $(-1)^n$ $-1, 1, -1, 1, \ldots$ DIV

If $r<-1$ r^n DIV

Def^n $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \ge 1$

$\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \ge 1$

$\{a_n\}$ is monotonic if it is either inc. or dec.

Example $\left\{\frac{3}{n+5}\right\}$ is decreasing

$$a_n = \frac{3}{n+5} > a_{n+1} = \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

Def^n $\{a_n\}$ is bounded above if $a_n \le M$ for all $n \ge 1$

$\{a_n\}$ is bounded below if $m \le a_n$ for all $n \ge 1$

$\{a_n\}$ is bounded if it is both bdd above and bdd below $\to$ bounded.

$$0 \le \frac{3}{n+5} \le a_1 = \frac{3}{1+5} = \frac{3}{6} = \frac{1}{2}$$

bdd below, also bdd above thus it is a bounded sequence.

Theorem (Monotonic Sequence Theorem) Every bounded, monotonic sequence is convergent.

11.2 Series

$\pi = 3.14159\ldots$

$\{a_n\}$ $\left\{3,\ 0.1 = \frac{1}{10},\ 0.04 = \frac{4}{10^2},\ \frac{1}{10^3},\ \frac{5}{10^4},\ \frac{9}{10^5}, \ldots\right\}$

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \cdots$$

$= a_1 + a_2 + a_3 + \cdots$ $\leftarrow$ Series.

Notation $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

e.g. $a_n = n$ $\quad n \ge 1$ $\quad 1, 2, 3, \ldots$

$$\sum_{n=1}^{\infty} a_n = 1+2+3+\cdots \qquad \infty$$

$$a_n = \frac{1}{2^n} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

Given a sequence a_n, to compute the series $\sum a_n$, we consider the partial sums

$a_1, a_2, a_3, a_4, \ldots$

partial sums:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

More generally, the n^{th} partial sum, denoted s_n, is $s_n = a_1 + a_2 + a_3 + \cdots + a_n$

Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent an we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series.

If $\{s_n\}$ is divergent then the series is also called divergent.

Example Suppose we are given that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

then

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{\frac{1}{n}(2n)}{\frac{1}{n}(3n+5)}$$

$$= \lim_{n\to\infty} \frac{2}{3+\underbrace{\frac{5}{n}}_{0}} = \frac{2}{3}$$

Example Consider the series given by

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

r is called the common ratio.

If $a=0$, $\sum_{n=1}^{\infty} ar^{n-1} = 0$. Assume $a \neq 0$.

If $r=1$ then $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$= a + a + \cdots + a = an$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} an = \pm\infty$ DIV

If $r>1$, $s_n = a + ar + \cdots + ar^{n-1}$

$\lim_{n\to\infty} s_n$ DIV.

If $-1<r<1$, $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$

So $s_n - rs_n = a - ar^n$

$s_n(1-r) = a(1-r^n)$ $\quad s_n = \frac{a(1-r^n)}{1-r}$

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-\cancel{r^n}^{\,0})}{1-r} = \frac{a}{1-r}$$

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is CONV only when $-1<r<1$ (or $a=0$)

and if it CONV the sum is $\frac{a}{1-r}$

"Geometric Series"