Arc Length

$r = f(\theta)$

$$\int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\sin\theta\cos\theta\frac{dr}{d\theta} + r^2\sin^2\theta$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\sin\theta\cos\theta\frac{dr}{d\theta} + r^2\cos^2\theta$$

so $$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) + 0 + r^2(\cos^2\theta+\sin^2\theta) = \underline{\underline{\left(\frac{dr}{d\theta}\right)^2 + r^2}}$$

$$\int ds = \int\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

__Example__ On the interval $[0,12]$, find the arc length of $r = 8\theta^2$.

$$\frac{dr}{d\theta} = 16\theta$$

$$L = \int_0^{12}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{12}\sqrt{64\theta^4 + (16\theta)^2}\, d\theta = \int_0^{12}\sqrt{64\theta^4 + 256\theta^2}\, d\theta$$

$$= \int_0^{12}\sqrt{256\theta^2\left(\frac{64}{256}\theta^2 + 1\right)}\, d\theta = \int_0^{12} \underline{16\theta}\sqrt{1 + \frac{1}{4}\theta^2}\, \underline{d\theta}$$

$u = 1 + \frac{1}{4}\theta^2$

$du = \frac{1}{2}\theta\, d\theta$

$2\,du = \theta\, d\theta$

$$= 32\int_a^b \underbrace{\sqrt{u}}_{u^{1/2}}\, du = 32\left(\frac{2u^{3/2}}{3}\right)\Big|_a^b$$

$$= \frac{64}{3}\left(1 + \frac{1}{4}\theta^2\right)^{3/2}\Big|_0^{12} = \frac{64}{3}\left(1 + \frac{1}{4}(12)^2\right)^{3/2} - \frac{64}{3}$$

11 Infinite Sequences and Series

__Sequence__: list of numbers

n^{th} term (the general term)

$a_{-1}, a_0, \cancel{a_1}, a_2, a_3, \ldots, a_n, \ldots$

__Notation__: $\{a_1, a_2, a_3, \ldots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

__Examples__

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \frac{n}{n+1}$ $\quad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

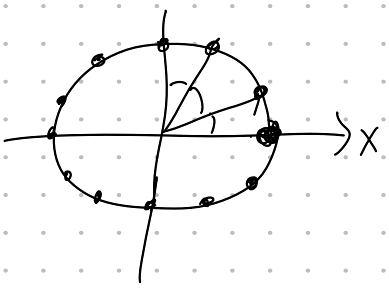
b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$ $\quad a_n = \frac{(-1)^n(n+1)}{3^n}$ $\quad \left\{\frac{(-1)(2)}{3}, \frac{(-1)^2 3}{3^2}, \frac{(-1)^3(4)}{3^3}, \ldots\right\}$

$$= \left\{-\frac{2}{3}, \frac{1}{3}, \frac{-4}{27}, \ldots\right\}$$

c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3} \quad n \geq 3$

$\{\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots\}$

d) $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\quad a_n = \cos\frac{n\pi}{6} \quad n \geq 0$ $\quad \{\underbrace{\cos 0}_{1}, \underbrace{\cos\frac{\pi}{6}}_{\sqrt{3}/2}, \underbrace{\cos\frac{2\pi}{6}}_{1/2},$



Fibonacci sequence

We define it recursively by the conditions

$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

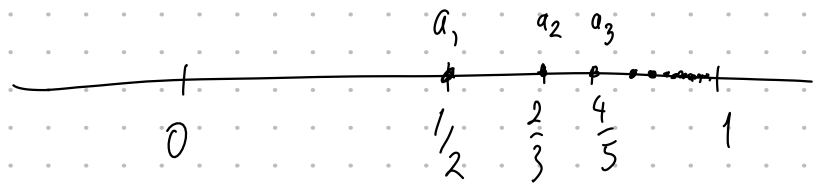
$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$\{1, 1, 2, 3, 5, 8, 13, 21 \ldots\}$

Plotting a sequence $a_n = \frac{n}{n+1}$ $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \dots \right\}$

I



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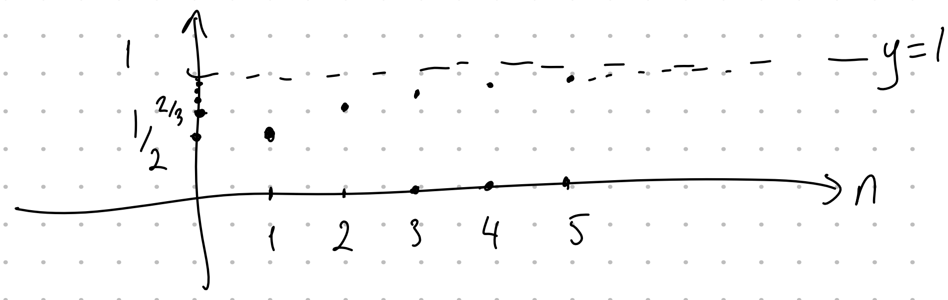


$\boxed{f(n) = a_n}$

n is a positive integer

$$\lim_{n\to\infty} a_n = 1$$

Definition A sequence $\{a_n\}$ has the Limit L and we write

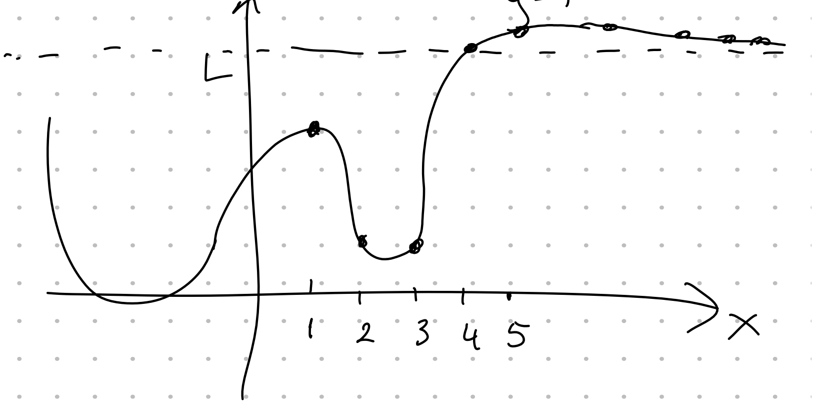
$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim\limits_{n\to\infty} a_n$ exists, we say $\{a_n\}$ is convergent.

Otherwise, it is divergent.

Thm If $\lim\limits_{x\to\infty} f(x) = L$ and $a_n = f(n)$ when n is an integer then $\lim\limits_{n\to\infty} a_n = L$.

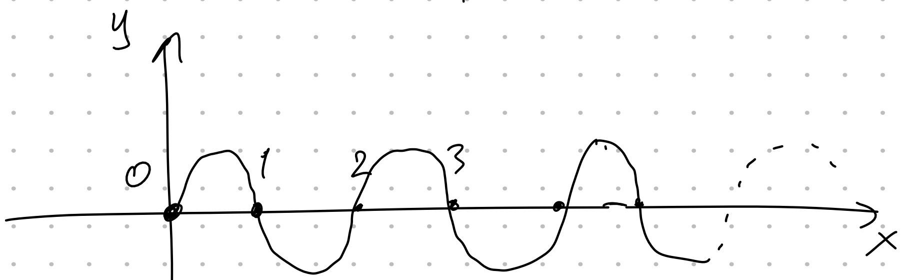


Converse is not true

$a_n = 0$

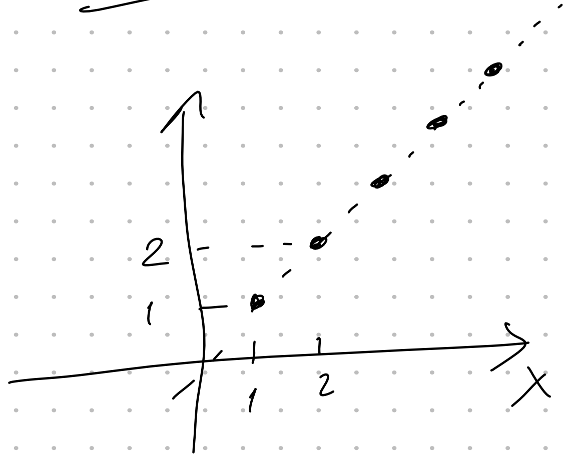
$f(x) = \sin(\pi x)$

So $a_n = 0 = \sin(\pi n) = f(n)$ $\lim\limits_{x\to\infty} f(x)$ DNE



Example $a_n = n$ $\{1, 2, 3, 4 \dots\}$

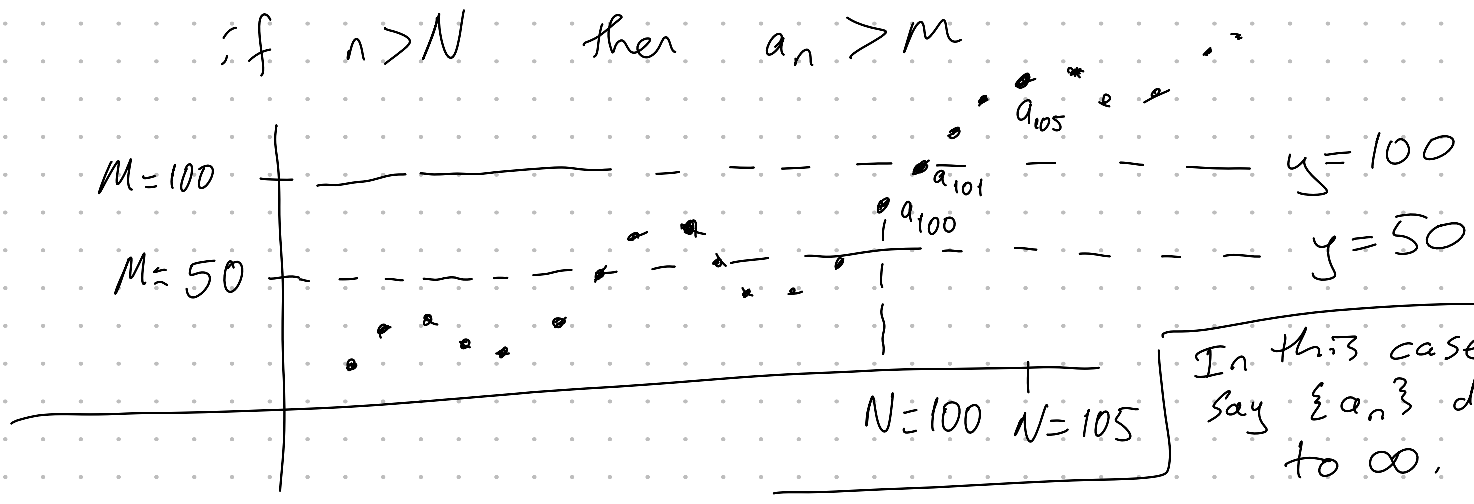
$$\lim_{n\to\infty} a_n = \infty$$



Definition $\lim\limits_{n\to\infty} a_n = \infty$ means that

for every positive number M there is an integer N such that

if $n > N$ then $a_n > M$



In this case we say $\{a_n\}$ diverges to ∞.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant then

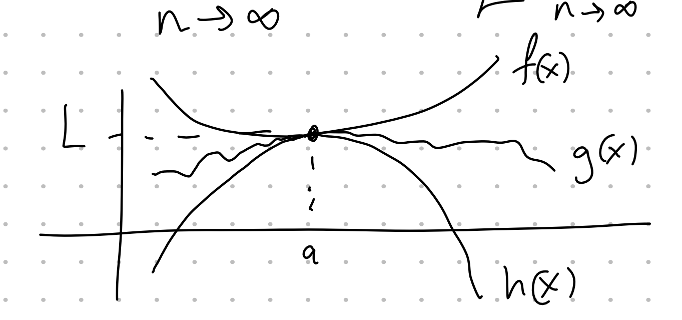
$$\lim_{n\to\infty} (a_n \pm b_n) = \left(\lim_{n\to\infty} a_n\right) \pm \left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{provided } \lim_{n\to\infty} b_n \neq 0$$

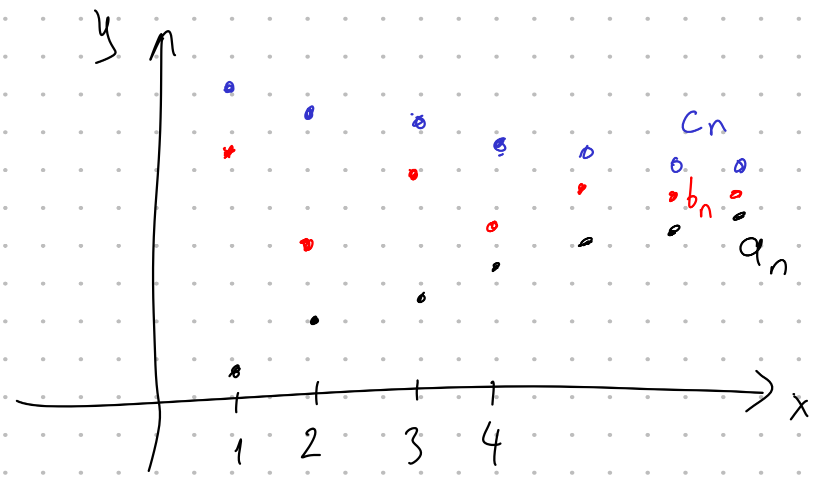
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$



$h(x) \leq g(x) \leq f(x)$ for all x

and $\lim\limits_{x\to a} h(x) = L = \lim\limits_{x\to a} f(x)$

then $\lim\limits_{x\to a} g(x) = L$ "The usual Squeeze Theorem"



$a_n \le b_n \le c_n$ for all n

and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

then $\lim_{n\to\infty} b_n = L$.