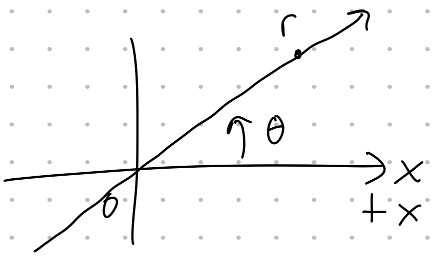
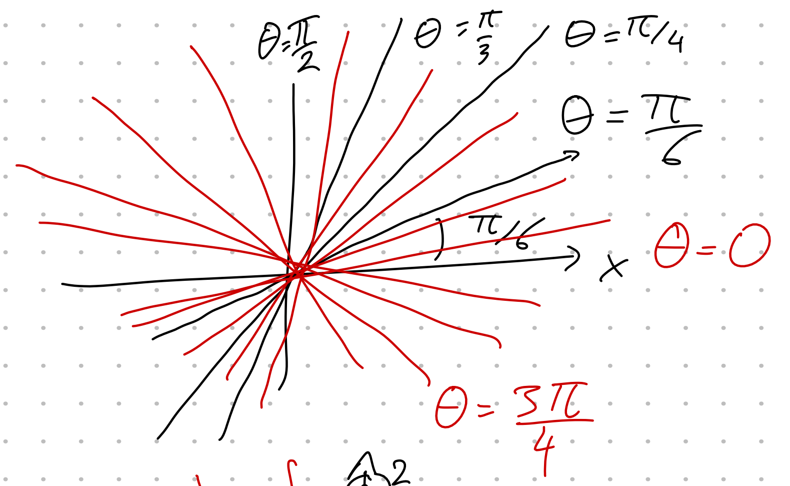
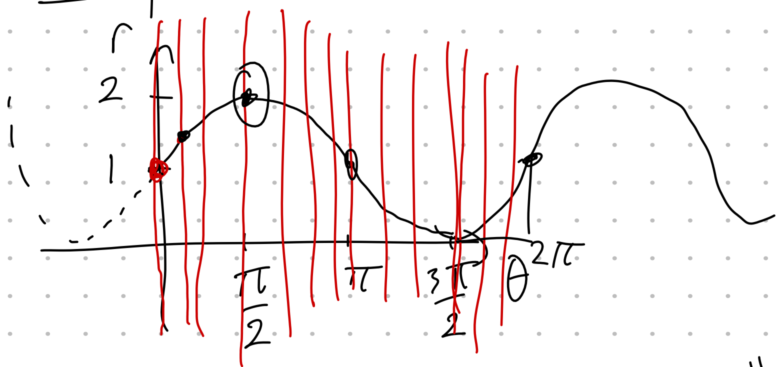
Polar Coordinates

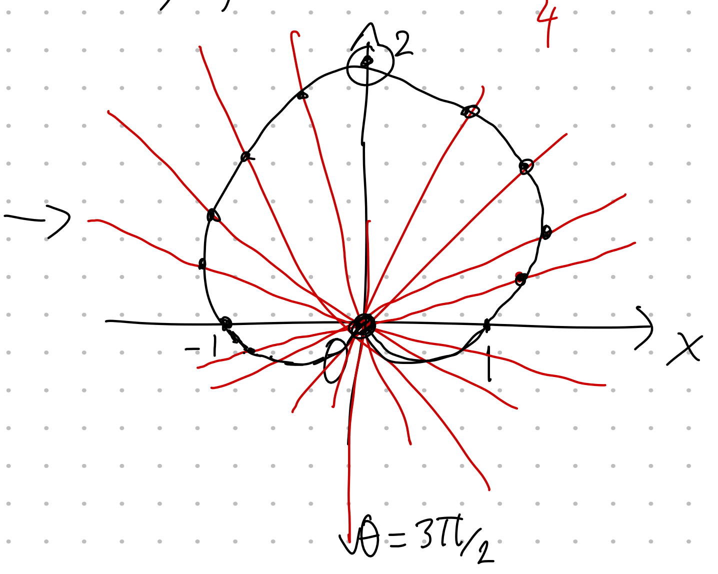
(r, θ)

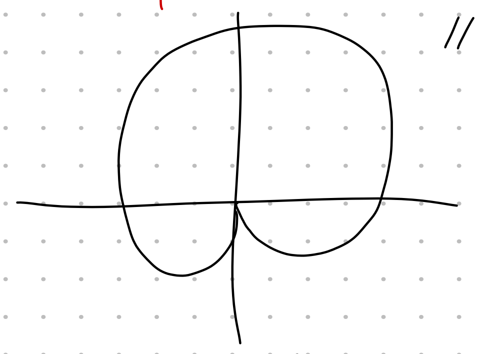




Example Sketch $r = 1 + \sin\theta$

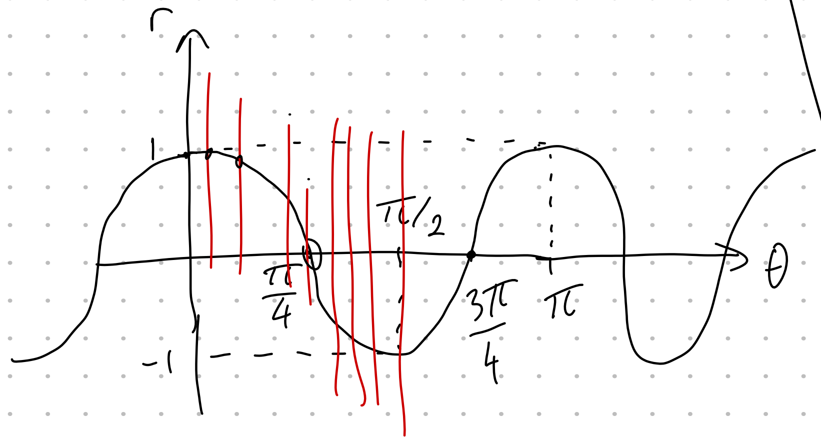


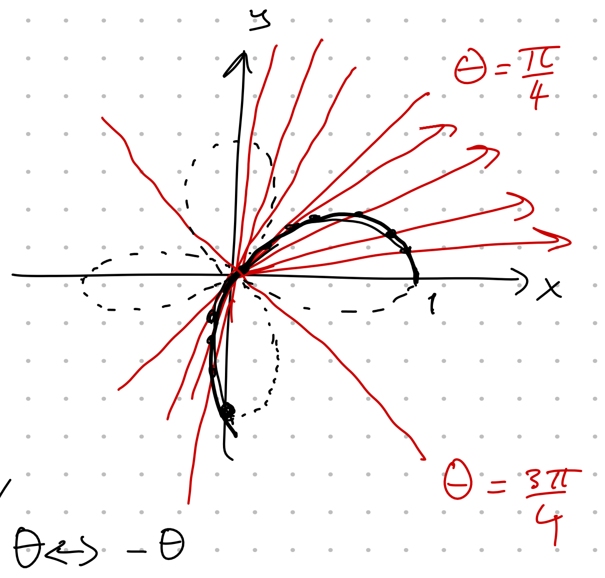


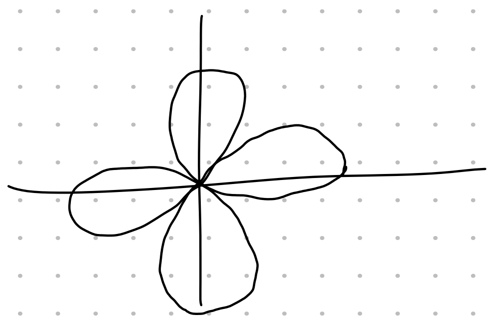


"Cardioid"

Example Sketch $r = \cos 2\theta$







a) $\theta \Leftrightarrow -\theta$

$r = \cos(2\theta) = \cos(-2\theta)$

b) $\theta \Leftrightarrow \theta + \pi$

$r = \cos(2(\theta+\pi)) = \cos(2\theta)$

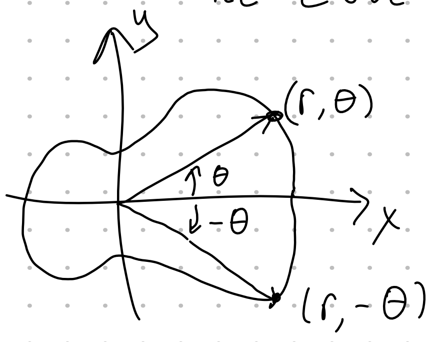
c) $r = \cos(2(\pi - \theta)) = \cos(2\theta)$

$\theta \Leftrightarrow \pi - \theta$

Symmetry $r = f(\theta)$

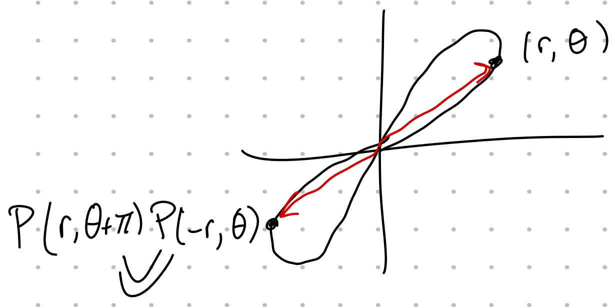
a) The equation does not change if $\theta \Leftrightarrow -\theta$

The curve is symmetric about the polar axis. (x-axis)



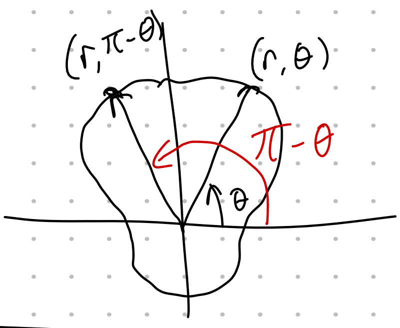
b) The equation does not change if $r \Leftrightarrow -r$ or $\theta \Leftrightarrow \theta + \pi$

The curve is symmetric about the pole



c) The equation does not change if $\theta \Leftrightarrow \pi - \theta$

The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ (y-axis)



Tangents to Polar Curves

$r = f(\theta)$ $\quad x = r\cos\theta \quad y = r\sin\theta$

$x = f(\theta)\cos\theta \quad y = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$\frac{dy}{dx} = 0$ "horizontal"

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{0}{c}$$

$\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$ horizontal

$\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$ vertical

$$\frac{dy}{dx} = \frac{c}{0}$$

Example a) For $r = 1+\sin\theta$ find the slope of the tangent line when $\theta = \pi/3$

b) Find the points where the tangent line is horizontal or vertical.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$y = r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta$

$x = r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \sin\theta\cos\theta$

$\frac{dy}{d\theta} = \cos\theta + 2\sin\theta\cos\theta = \cos\theta(1+2\sin\theta)$

$\frac{dx}{d\theta} = -\sin\theta + \cos^2\theta - \sin^2\theta = 1 - \sin\theta - 2\sin^2\theta$

$= (1+\sin\theta)(1-2\sin\theta)$

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1-\sin^2\theta$

ⓐ $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}$

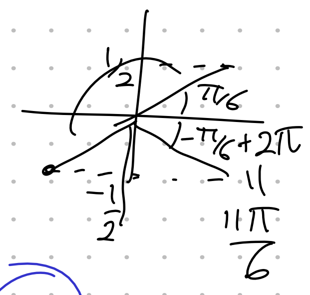
$$= \frac{\cos\frac{\pi}{3}\left(1+2\sin\frac{\pi}{3}\right)}{\left(1+\sin\frac{\pi}{3}\right)\left(1-2\sin\frac{\pi}{3}\right)}$$

$$= \frac{\frac{1}{2}\left(1+\cancel{2}\frac{\sqrt3}{\cancel{2}}\right)}{\left(1+\frac{\sqrt3}{2}\right)\left(1-\cancel{2}\frac{\sqrt3}{\cancel{2}}\right)} = \frac{1+\sqrt3}{2\left(1+\frac{\sqrt3}{2}\right)(1-\sqrt3)} = \frac{1+\sqrt3}{(2+\sqrt3)(1-\sqrt3)} = \frac{1+\sqrt3}{2-2\sqrt3+\sqrt3-3}$$

$$= \frac{1+\sqrt3}{-1-\sqrt3} = -1$$

ⓑ If $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$ → horizontal

If $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$ → vertical



$\frac{dy}{dx} =$

$\frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0$

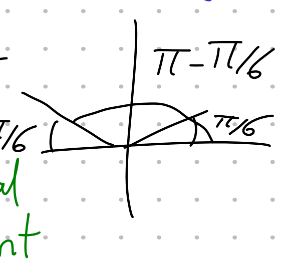
$\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \pi + \frac{\pi}{6} = \frac{7\pi}{6}, \frac{11\pi}{6}$ Horizontal Tangent

$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0$

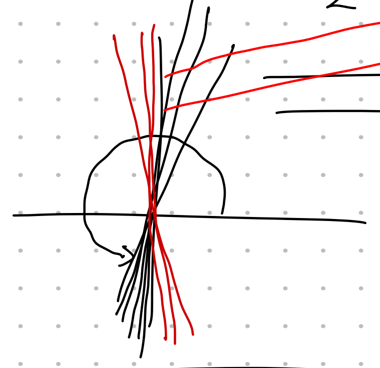
$\sin\theta = -1$ or $\sin\theta = \frac{1}{2}$

$\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ Vertical Tangent



$$\lim_{\theta\to\frac{3\pi}{2}^+} \frac{dy}{dx} = \lim_{\theta\to\frac{3\pi}{2}^+} \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)} = \left(\lim_{\theta\to\frac{3\pi}{2}^+} \frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to\frac{3\pi}{2}^+}\frac{\cos\theta}{1+\sin\theta}\right)$$

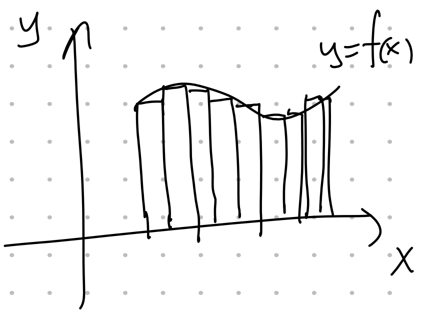
$$= \frac{1+2(-1)}{1-2(-1)} \lim_{\theta\to\frac{3\pi}{2}^+} \frac{-\sin\theta}{\cos\theta} = \frac{+1}{3} \lim_{\theta\to\frac{3\pi}{2}^+} \tan\theta = -\infty$$



$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{dy}{dx} = +\infty$$

at $\theta = \frac{3\pi}{2}$ we have a vertical tangent!

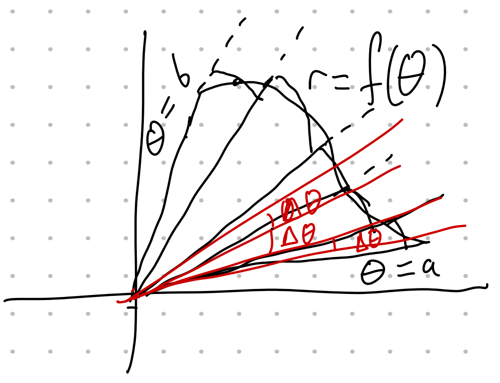
10.4 Areas and Lengths in Polar Coordinates



Wedge "a sector of a circle"

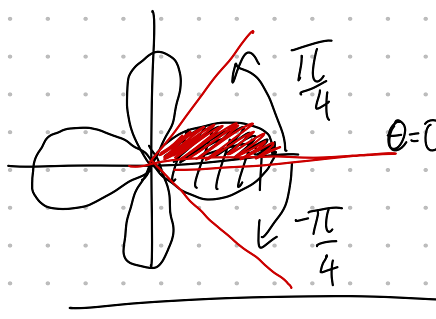
Area $= \frac{1}{2} r^2\theta$

$\theta = 2\pi \rightarrow A = \frac{\cancel{2}\pi r^2}{\cancel{2}}$ ✓



$$\text{Area} = \int_a^b \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_a^b (f(\theta))^2\, d\theta$$

Example Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



$$\text{Area} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta = \frac{\cancel{1}}{\cancel{2}}\cancel{2}\int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$\boxed{\cos^2 x = \frac{1}{2}(1+\cos 2x) \quad x = 2\theta}$$

$$= \frac{1}{2}\int_0^{\pi/4} (1+\cos 4\theta)\, d\theta = \frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{4}\cancel{\sin\left(\frac{4\pi}{4}\right)}^{\,0}\right) - 0 = \frac{\pi}{8}$$