Last time: Parametric curves

$x = f(t) \quad y = g(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{d^2x}{dt^2}}$$

Example A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$

a) Show that C has two tangents at the point $(3,0)$ and find their equation.

Set $x = 3,\ y = 0$

or $t^2 = 3,\ t^3 - 3t = 0$

$t = 0$ satisfies the second equation however, it does not satisfy the first one. So $t = 0$ is not a common solution.

from $t^2 = 3 \rightarrow t = \pm\sqrt{3}$

and $(\pm\sqrt{3})^3 - 3(\pm\sqrt{3}) = 0$ ✓

So the solutions are $t = \pm\sqrt{3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} \quad \text{so}$$

$$\left.\frac{dy}{dx}\right|_{t=\pm\sqrt{3}} = \frac{3(\pm\sqrt{3})^2 - 3}{\pm 2\sqrt{3}} = \pm\frac{\cancel{6}\ ^{3}}{\cancel{2}\sqrt{3}}\ ^{\sqrt{3}}$$

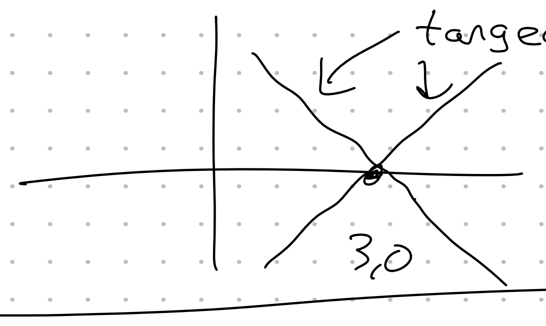
so at $t = \sqrt{3}$ $m = \sqrt{3}$ $\quad (x_0, y_0) = (3,0)$

$t = -\sqrt{3}$ $m = -\sqrt{3}$

$y - y_0 = m(x - x_0)$

$y - 0 = \sqrt{3}(x - 3) \rightarrow y = \sqrt{3}x - 3\sqrt{3}$

$y - 0 = -\sqrt{3}(x - 3) \rightarrow y = -\sqrt{3}x + 3\sqrt{3}$



b) Find the points on C where the tangent is horizontal or vertical.

The tangent line is horizontal if $\frac{dy}{dx} = 0$.

So we set $\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = 0$

So $3t^2 - 3 = 0 \qquad t^2 = 1 \qquad t = \pm 1$

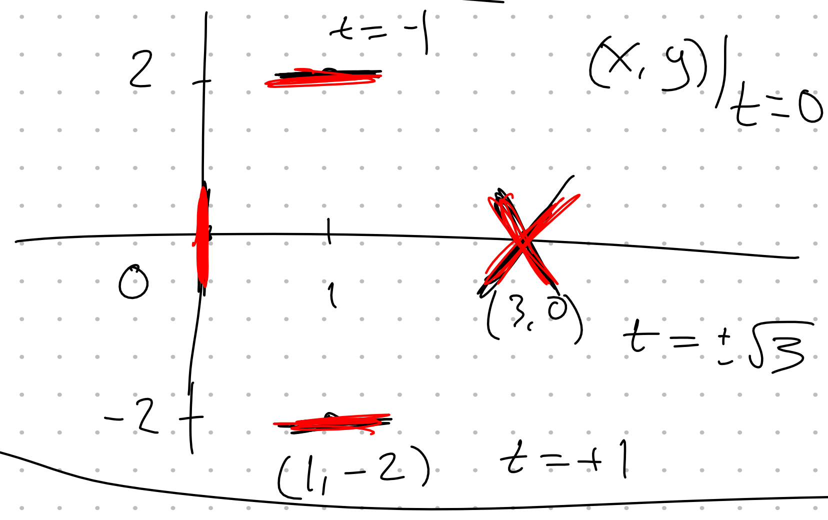
So $(x,y) = (t^2, t^3 - 3t)\big|_{t=\pm 1} = (1, \pm 1 - 3(\pm 1))$

$= (1, \pm 1 \mp 3) = (1, \mp 2)$

The tangent line is vertical if $\frac{dy}{dx}$ has a vertical asymptote.

$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$ so $t = 0$ is the only vert. asymp.

$(x,y)\big|_{t=0} = (0,0)$



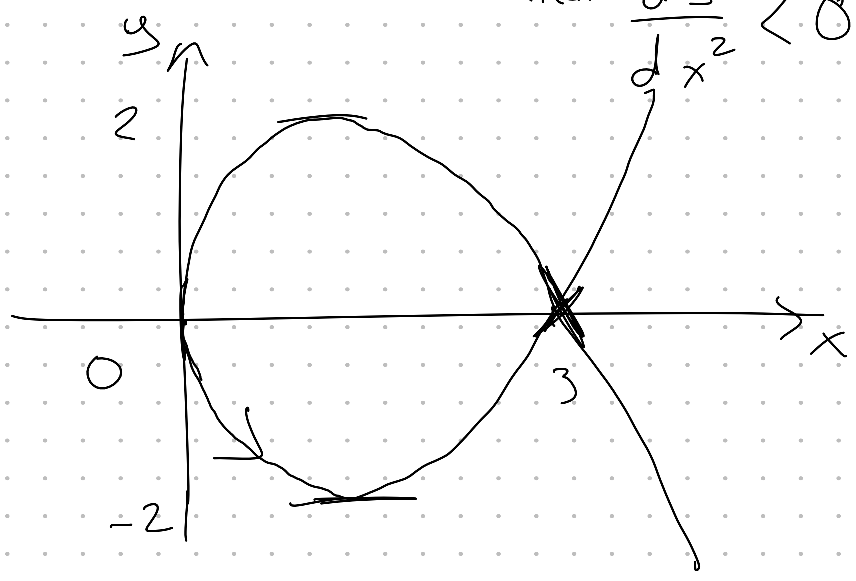
c) Determine where C is conc. up or down.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$= \frac{\frac{6t(2t) - (3t^2-3)2}{(4t^2)}}{2t} = \frac{12t^2 - 6t^2 + 6}{8t^3}$$

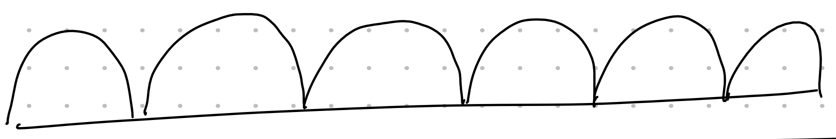
$$= \frac{6(t^2+1)}{8t^3} \quad \text{so} \quad \frac{d^2y}{dx^2} > 0 \text{ if } t > 0$$

and $\frac{d^2y}{dx^2} < 0$ if $t < 0$



Cycloid: $x = r(\theta - \sin\theta)$
$y = r(1 - \cos\theta)$

If we scale it appropriately



Exaple

a) Find the tangent to the cycloid at the point where $\theta = \pi/3$.

$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\sin\theta}{r - r\cos\theta} = \frac{\cancel{r}\sin\theta}{\cancel{r}(1-\cos\theta)}$$

$$m\big|_{\theta=\pi/3} = \frac{\sin \pi/3}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$(x,y)\big|_{\theta=\frac{\pi}{3}} = \left(r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right), r\left(1 - \cos\frac{\pi}{3}\right)\right)$$

$$= \left(r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \frac{r}{2}\right) \qquad y - y_0 = m(x - x_0)$$

$$y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

b) At what points is the tangent horizontal? When is it vertical?

Set $\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = 0$ so $\sin\theta = 0$

$\theta = 0, \pi, 2\pi, 3\pi, \ldots$ $\quad \theta = k\pi$ for any integer k.

$-\pi, -2\pi, \ldots$

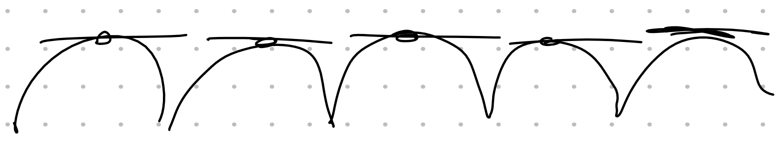
$$(x,y)\big|_{\theta=k\pi} = (r(k\pi - \underbrace{\sin k\pi}_{0}), r(1 - \underbrace{\cos(k\pi)}_{(\pm 1)}))$$

However, when $1 - \cos\theta = 0$ $\frac{dy}{dx}$ is not defined. So we should exclude the points where

$1-\cos\theta = 0$ or $\cos\theta = 1$. So $\theta = 2\pi k$ is not allowed!

So we have horizontal tangent at $\theta = k\pi$ for only odd integers k.

$$(x,y)\Big|_{\theta=k\pi} = \left(rk\pi,\ r(1-(-1))\right) = (kr\pi, 2r)$$

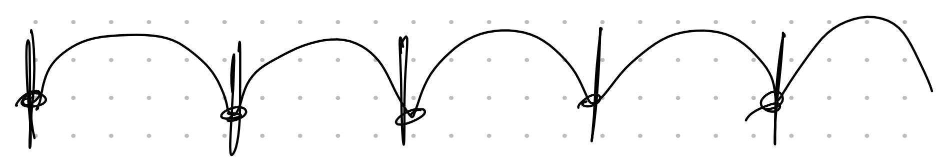


$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$ has a vertical asymp. when $1-\cos\theta = 0$

$\theta = 2\pi k$ for any integer k.

$$(x,y)\Big|_{\theta=2\pi k} = \left(r(2\pi k - \sin(2\pi k)),\ r(1-\cos 2\pi k)\right) = (2kr\pi, 0)$$

for any integer k.



<u>Areas</u> for $y = F(x)$ the area under the curve is

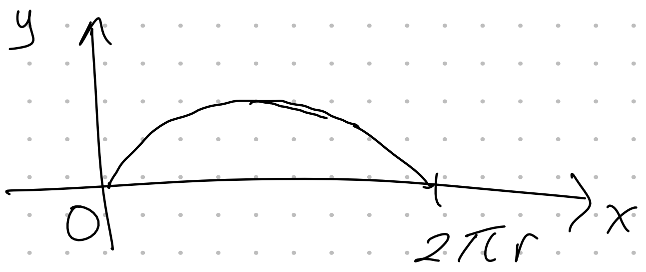
$$\int_a^b F(x)\,dx = \int_a^b y\,dx$$

If, instead, we have a parametric curve $x = f(t)$, $y = g(t)$, $dx = f'(t)\,dt$ so

$$\int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$$ where $x = a = f(\alpha)$, $x = b = f(\beta)$

<u>Example</u> Find the area under one arch of the cycloid.

$x = r(\theta - \sin\theta)$ $\quad y = r(1-\cos\theta)$



$$\int_0^{2\pi r} y\,dx$$

$x = 0$ when $\theta = 0$

$x = 2\pi r$ when $\theta = 2\pi$

$dx = r(1-\cos\theta)\,d\theta$ so

$$\int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$

$$= r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= r^2\left(\theta\Big|_0^{2\pi} - 0 + \frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta\right)$$

$$= r^2\left(2\pi + \frac{1}{2}\,\theta\Big|_0^{2\pi} + 0\right) = 3\pi r^2$$

<u>Arc length</u> For $y = F(x)$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$\int \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt$$

$$\int \sqrt{\left(1 + \left(\frac{y'}{x'}\right)^2\right)}\,x'\,dt$$

$y' = \frac{dy}{dt}$

$x' = \frac{dx}{dt}$

$$= \int \sqrt{\left(1+\left(\frac{y'}{x'}\right)^2\right)(x')^2}\,dt$$

$$= \int \sqrt{(x')^2 + (y')^2}\,dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$