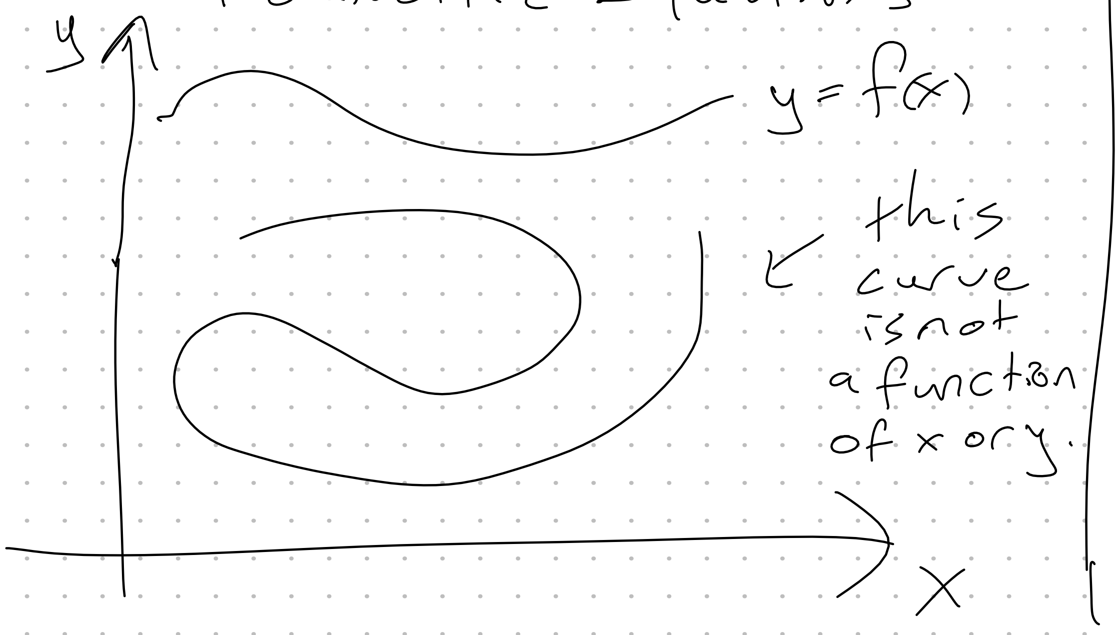
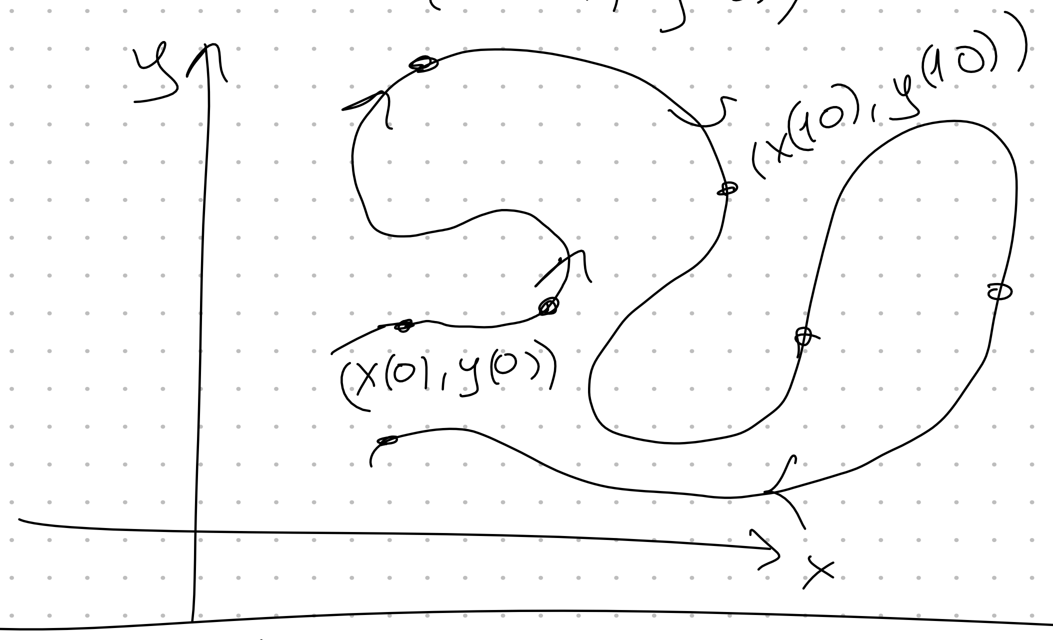
10.1 Curves defined by Parametric Equations



A particle moving in a plane

$(x(t), y(t))$



Suppose that both x and y are given as functions of a third variable (called a parameter) by the equations

$x = f(t)$ and $y = g(t)$

called parametric equations. In this case we call the set of points of the form $(x(t), y(t))$ a parametric curve.

Example Sketch and identify the curve defined by

$x = t^2 - 2t \qquad y = t + 1$

t	x	y
-2	$4+4=8$	-1
-1	$1+2=3$	0
0	0	1
1	$1-2=-1$	2
2	0	3

$t = y - 1$

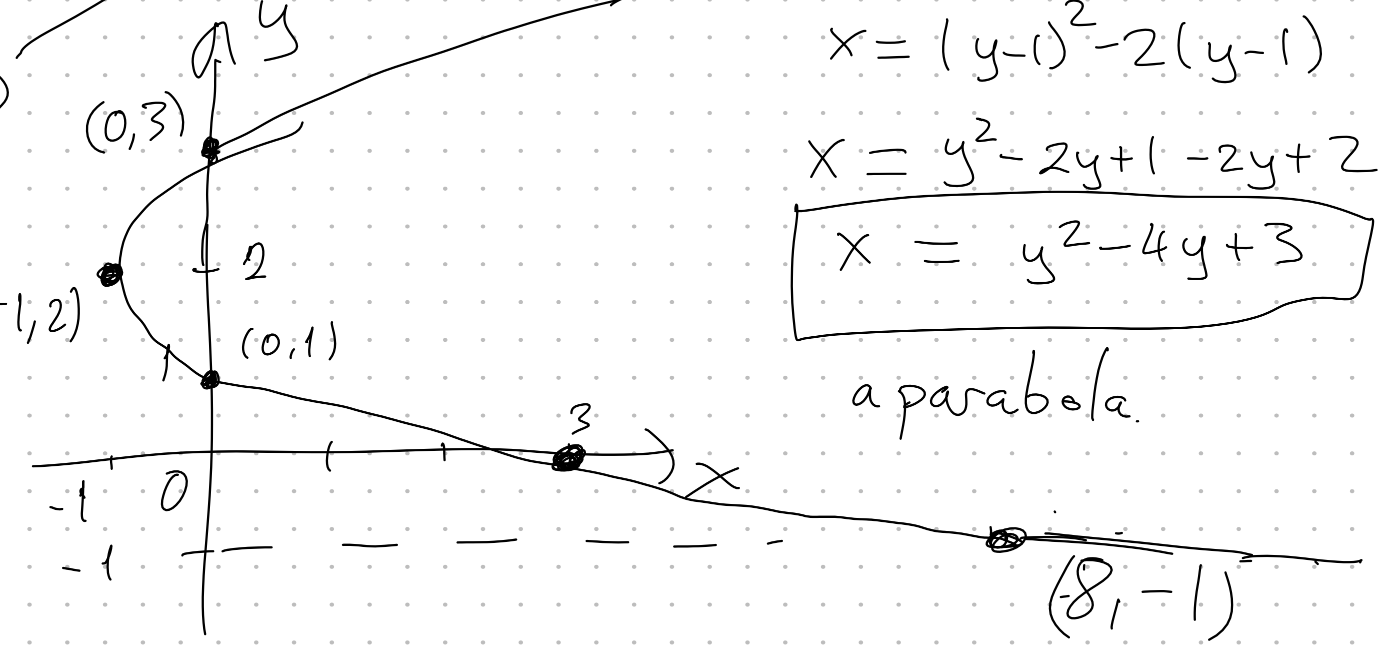
So

$x = (y-1)^2 - 2(y-1)$

$x = y^2 - 2y + 1 - 2y + 2$

$\boxed{x = y^2 - 4y + 3}$

a parabola

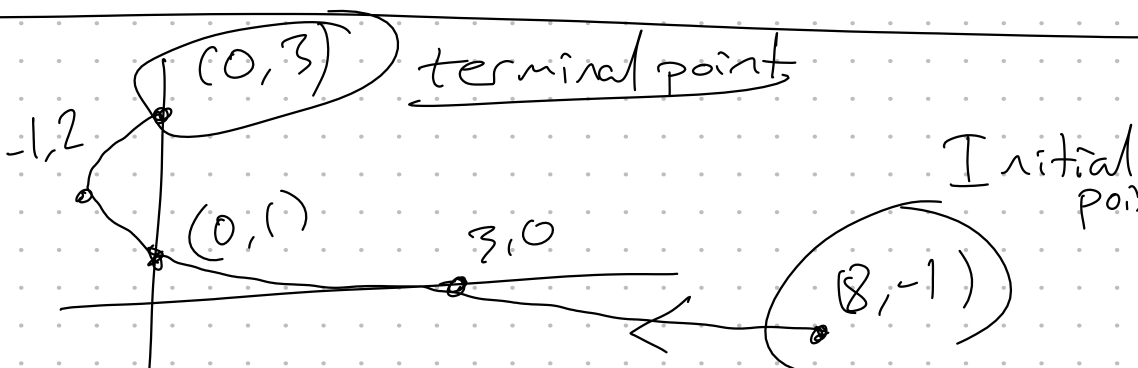


If we had a restricted domain for the example above, the equation would like

$x = t^2 - 2t \quad y = t + 1 \quad -2 \le t \le 2$

gives us the initial point

gives us the terminal point



In this case

$(f(-2), g(-2)) = (8,-1)$ is called the initial point

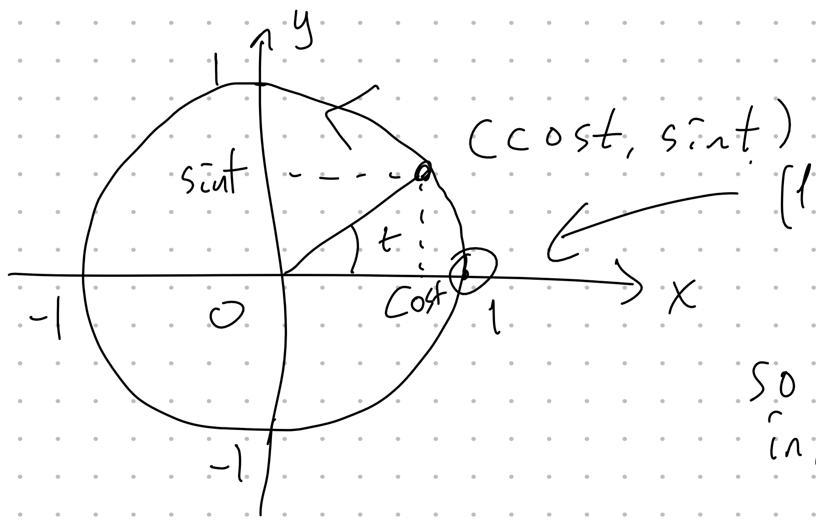
$(f(2), g(2))$ is the terminal point

Example What curve is represented by the following parametric equations?

$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

since $0 \le t \le 2\pi$ we get the whole unit circle.

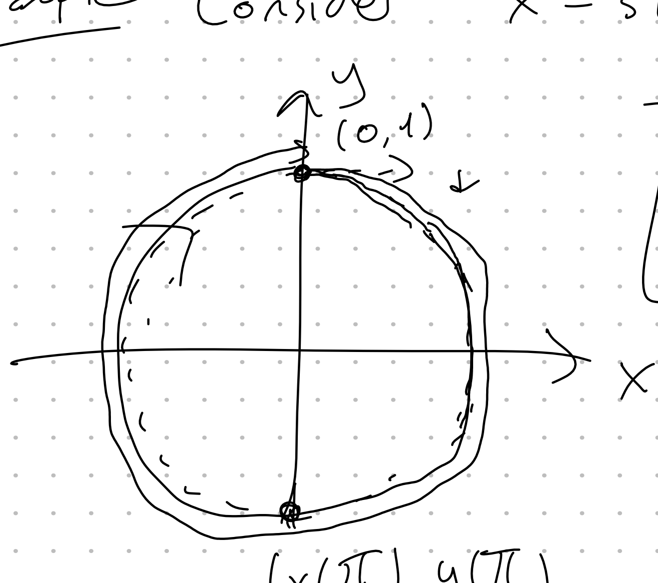


$(1,0) = (\cos 0, \sin 0)$
$= (\cos 2\pi, \sin 2\pi)$

so it is both the initial and the terminal point.

Also, the parametric curve is tracing the unit circle in the counter clockwise direction (CCW)

Example Consider $x = \sin 2t \quad y = \cos 2t \quad 0 \le t \le 2\pi$



$(x(\frac{\pi}{2}), y(\frac{\pi}{2}))$
$= (0,-1) = (x(\frac{3\pi}{2}), y(\frac{3\pi}{2}))$

$\boxed{(x(0), y(0)) = (0,1)}$

$\boxed{(x(\pi), y(\pi)) = (0,1)} = (x(2\pi), y(2\pi))$

again $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$

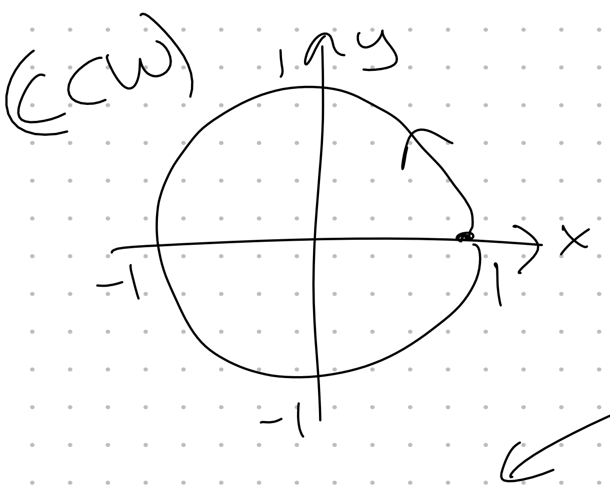
So the curve is a subset of the unit circle (possibly the whole circle.)

So again the initial and terminal points are the same $(0,1)$

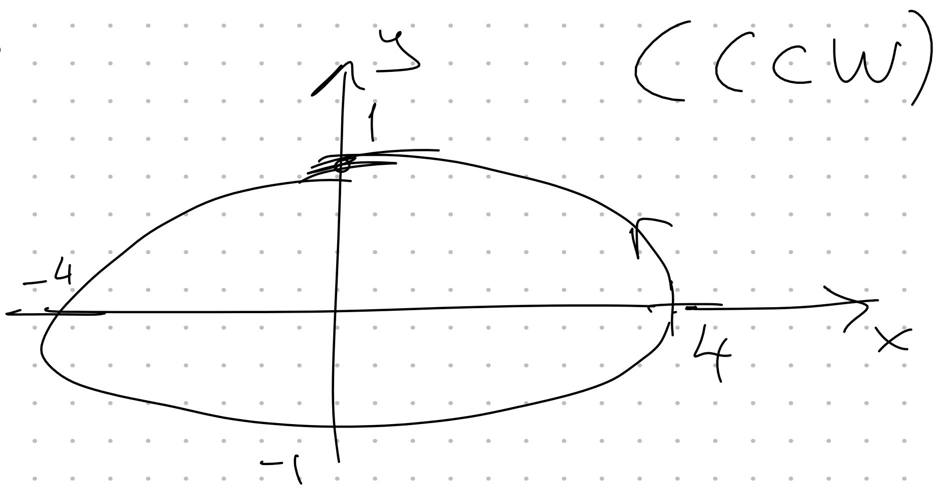
The main difference is we are traversing the unit circle twice in clockwise direction. (CW)

Example Consider $x = 4\cos t$, $y = -\sin t$

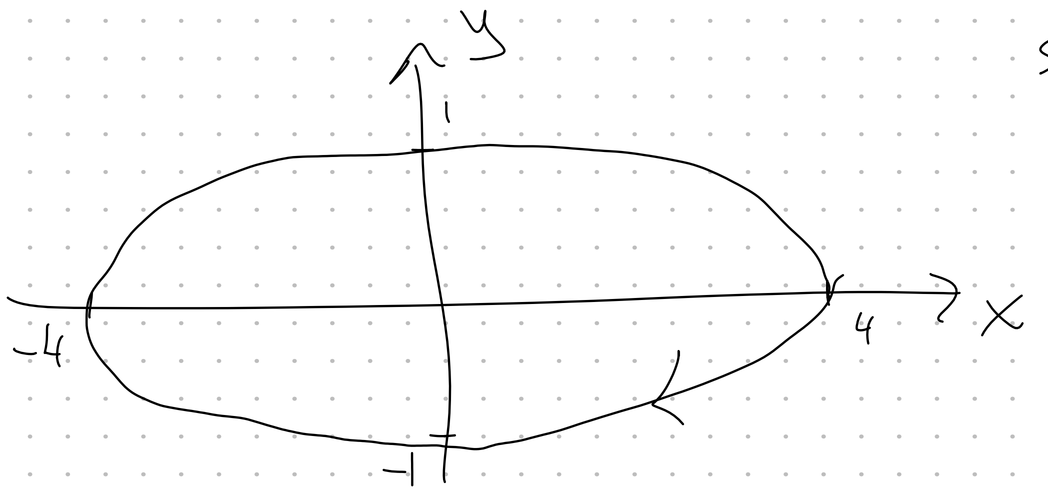
$x = \cos t$ $\quad y = \sin t$



multiply x by 4 $\longrightarrow$



multiply y by (-1) (this reflects the picture about x-axis)

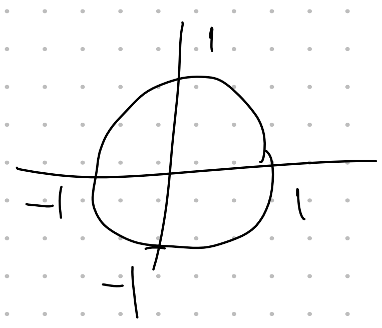


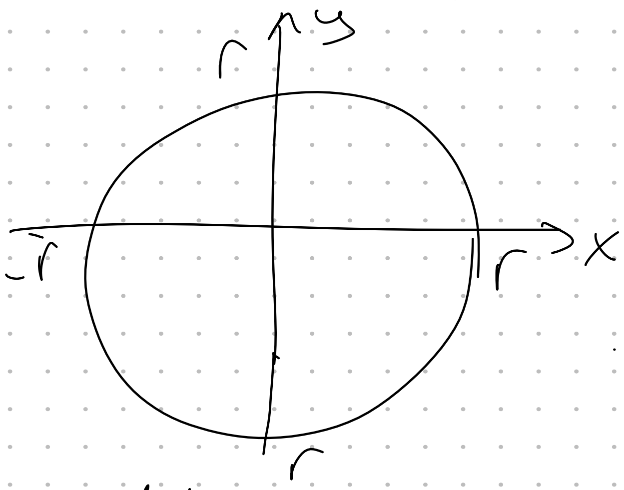
so the direction of the particle changes

(CW)

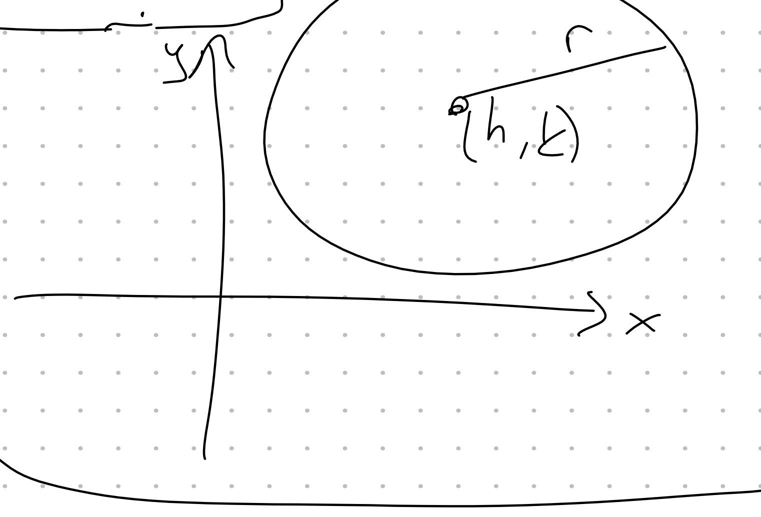
Example Find the parametric equations of a circle of radius r centered at (h,k).

Start with $x = \cos t$ $\quad y = \sin t$ $\longrightarrow$ $x = r\cos t$, $y = r\sin t$





to translate the center of the circle from $(0,0)$ to (h,k) we add h to x-coords. and k to y-coords. So $\boxed{x = r\cos t + h \quad y = r\sin t + k}$



10.2 Calculus with parametric curves.

Tangents

$\dfrac{dy}{dx} = ?$ when x and y are functions of t.

Assume $y = y(x(t))$ then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$ by the chain rule.

So if $\dfrac{dx}{dt} \neq 0$, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\quad$ If $\dfrac{dx}{dt} = 0$ the the tangent line is vertical if $\dfrac{dy}{dt} \neq 0$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \underset{\substack{\text{by the same reasoning}\\ \text{as above}}}{=} \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example Consider the curve C, $x = t^2$, $y = t^3 - 3t$

a) Show that C has two tangents at $(3,0)$ and find their equations

b) Find the points on C where the tangent is horizontal or vertical.

$x = t^2 = 3 \longrightarrow \boxed{t = \pm\sqrt{3}}$

$y = t^3 - 3t = 0 \longrightarrow \cancel{t}^{\,\cancel{3}\,2} = 3\cancel{t} \rightsquigarrow t^2 = 3$

In fact $t = 0$ solves $y = 0$ but since it does not solve $x = 3$, we reject $t = 0$ as we are looking for common solutions to $x = 3$ and $y = 0$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} \quad \text{so} \quad \left.\frac{dy}{dx}\right|_{t=\pm\sqrt{3}} = \frac{9-3}{\pm 2\sqrt{3}} = \pm\frac{\cancel{6}^{\,3}}{\cancel{2}\sqrt{3}} = \pm\sqrt{3}$$

So $m = \pm\sqrt{3}$ $(x_0, y_0) = (3, 0)$ so

$y - y_0 = m(x - x_0)$

$y - 0 = \sqrt{3}(x-3)$

$y - 0 = -\sqrt{3}(x-3)$

Equations of the tangent lines at $(3,0)$