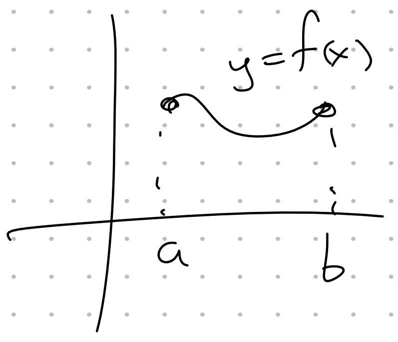
Last time:

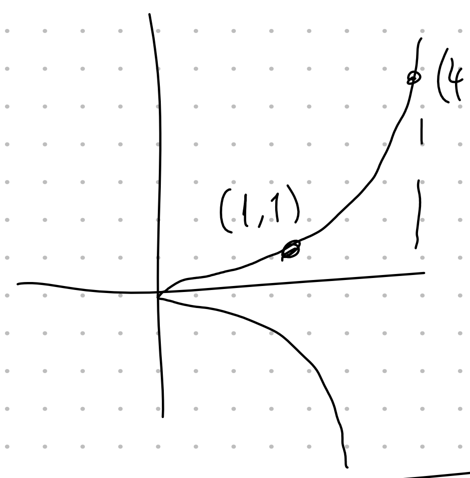
Arc length $= L$




$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example Find the length of the arc of $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$




$y = \pm\sqrt{x^3}$

$y = x^{3/2}$

$y' = \frac{3}{2}x^{1/2}$

So $(y')^2 = \frac{9}{4}x$

$$L = \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$u = 1+\frac{9}{4}x \quad du = \frac{9}{4}dx$

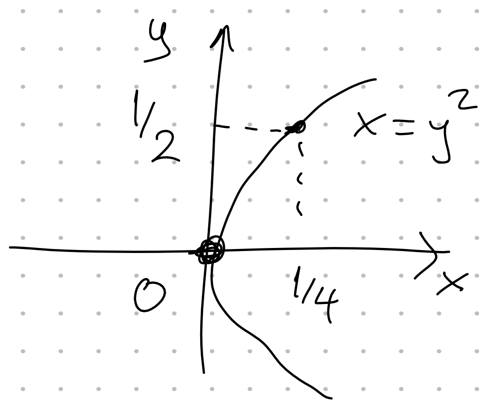
$x = 1 \quad u = 1+\frac{9}{4} = 13/4$

$x = 4 \quad u = 1+\frac{9}{4}\cdot 4 = 10$

$$\int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big|_{13/4}^{10}$$

$$= \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)$$

Example Set up an integral for the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $\left(\frac{1}{4},\frac{1}{2}\right)$.



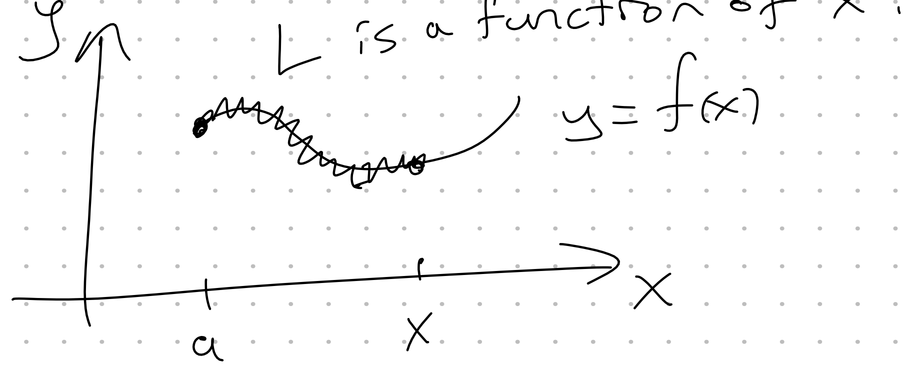

$x' = \frac{dx}{dy} = 2y$

$(x')^2 = 4y^2$

$$L = \int_0^{1/2} \sqrt{1+4y^2}\,dy$$

Arc Length Function

$L$ is a function of $x$.




We denote this dependence of $L$ on $x$ by

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt$$

arc length function.

$$\frac{ds}{dx} = \sqrt{1+(f'(x))^2} \quad \text{by FTC}$$

$$\geq 1$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $(1,1)$ as the starting point.

$y' = 2x - \frac{1}{8x} \qquad (y')^2 = \left(2x - \frac{1}{8x}\right)^2$

So $(y')^2 = 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$

So $1+(y')^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$

$$= \left(2x + \frac{1}{8x}\right)^2$$

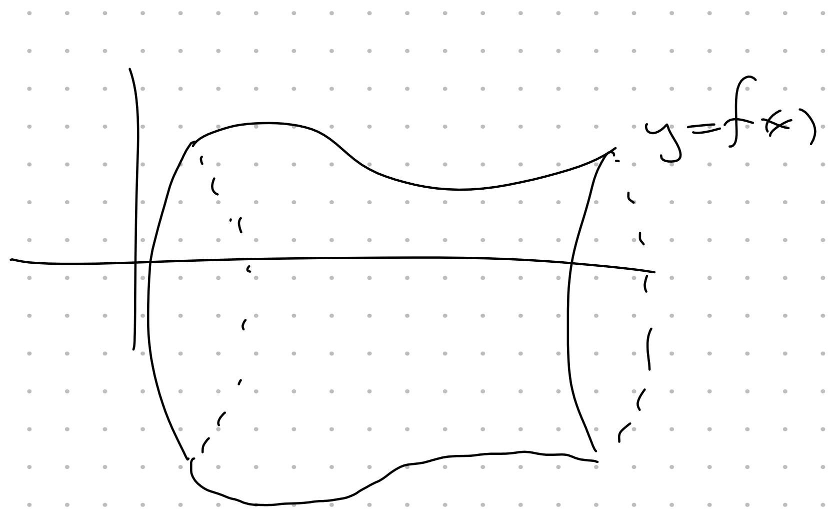
So

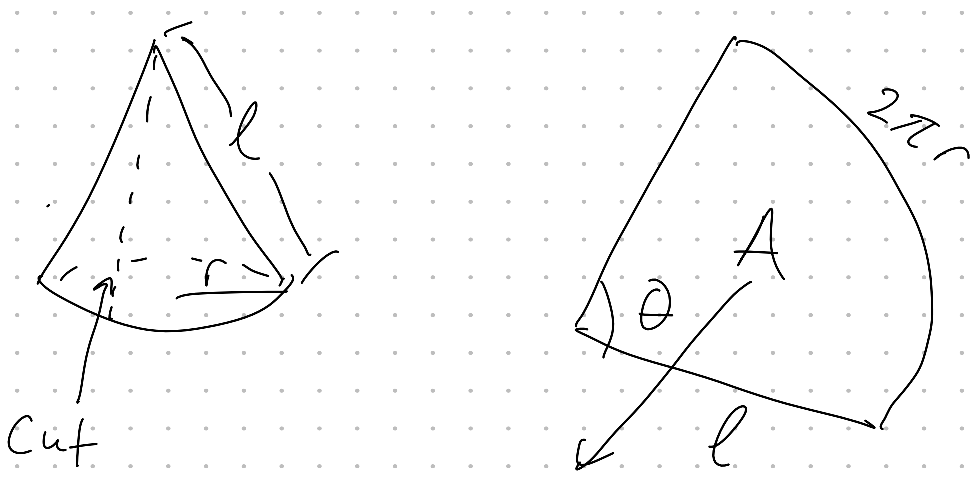
$$s(x) = \int_1^x \sqrt{1+(y')^2}\,dt = \int_1^x \sqrt{\left(2t+\frac{1}{8t}\right)^2}\,dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right)dt = t^2 + \frac{1}{8}\ln t\Big|_1^x$$

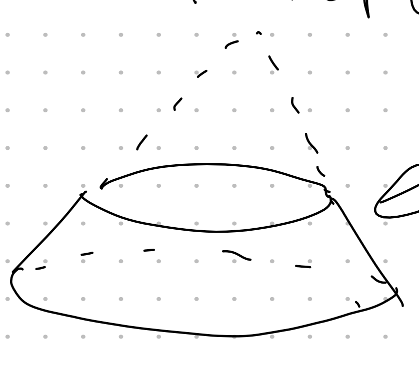
$$= x^2 + \frac{1}{8}\ln x - 1$$

## 8.2 Area of a Surface of Revolution



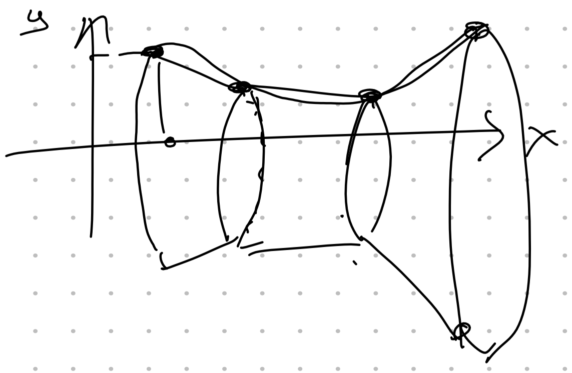




$$A = \pi r l = \frac{1}{2}(2\pi r)\,l$$



Area of this band is area of the big cone − area of the small cone.


$$A = \pi r_2 (l_1 + l) - \pi r_1 l_1$$
$$= 2\pi r l$$
where $r = \dfrac{r_1 + r_2}{2}$




We approximate the surface area by a number of bands.

So $S.A. \simeq \sum_{i=1}^{N} 2\pi r_i l_i$

In the limit;

$$S.A = \int_a^b 2\pi f(x) \overbrace{\sqrt{1+(f'(x))^2}\, dx}^{ds}$$

$ds$ length of a tiny line segment

$r$ is $f(x)$ since we rotate the curve about the $x$-axis.

So $$SA = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$SA = \int 2\pi y \, ds$$

If the curve is rotated about the $y$-axis then

$$SA = \int 2\pi x \, ds$$

where $ds = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx = \sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\, dy$

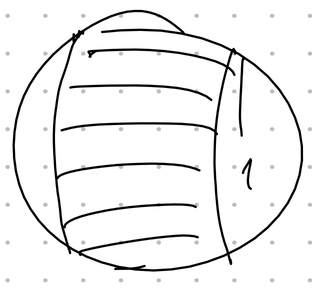
Example The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2+y^2=4$. Find the area of the surface obtained by rotating this arc about the $x$-axis.

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$$

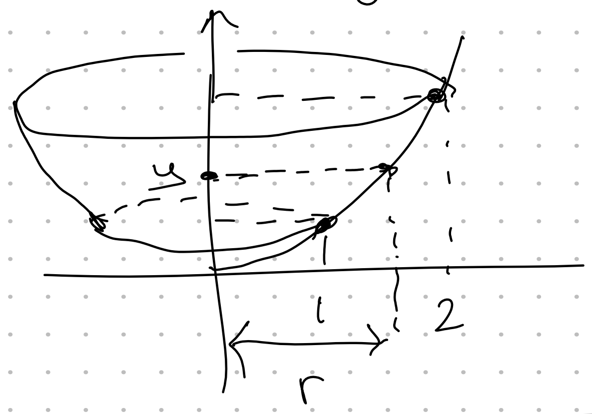
$$A = \int_{-1}^{1} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{-1}^{1} 2\pi \sqrt{4-x^2} \sqrt{1+\frac{x^2}{4-x^2}}\, dx$$

$$= \int_{-1}^{1} 2\pi \cancel{\sqrt{4-x^2}} \sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{\cancel{4-x^2}}}\, dx$$

$$= \int_{-1}^{1} 2\pi \sqrt{4}\, dx = 4\pi \int_{-1}^{1} dx = 8\pi$$




Example the arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$ is rotated about the $y$-axis. Find the area of the resulting surface




Sol 1

$r = x = \sqrt{y}$

so $\dfrac{dx}{dy} = \dfrac{1}{2\sqrt{y}}$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{4y} \qquad 2\pi r (ds)$$

$$A = \int_1^4 2\pi \sqrt{y} \sqrt{1+\frac{1}{4y}}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y+\frac{1}{4}}\, dy \qquad u = y + \frac{1}{4}, \quad du = dy$$

$$= 2\pi \int_{1+1/4}^{4+1/4} \sqrt{u}\, du \qquad y=4 \quad u = 4+\frac{1}{4}, \qquad y=1 \quad u = 1+\frac{1}{4}$$

$$= 2\pi \frac{2}{3} u^{3/2} \Big|_{1+1/4}^{4+1/4} = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

Sol 2

$$S = \int 2\pi x \, ds \qquad (y = x^2)$$

$$\frac{dy}{dx} = 2x$$

$$= \int_1^2 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \sqrt{1+4x^2}\, dx \qquad u = 1+4x^2, \quad du = 8x\,dx$$

$$= \frac{1}{8}(2\pi) \int \sqrt{u}\, du \qquad x=1 \Rightarrow u=5, \quad x=2 \Rightarrow u=17$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_5^{17} = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$