Last time: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \pi/2 + \pi/2 = \pi$$

Example: For what values of p is the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Recall from last time if $p=1$,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t \Big|_1^t$$

DIV

If $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t = \lim_{t\to\infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1^{(-p+1)}}{-p+1} \right)$$

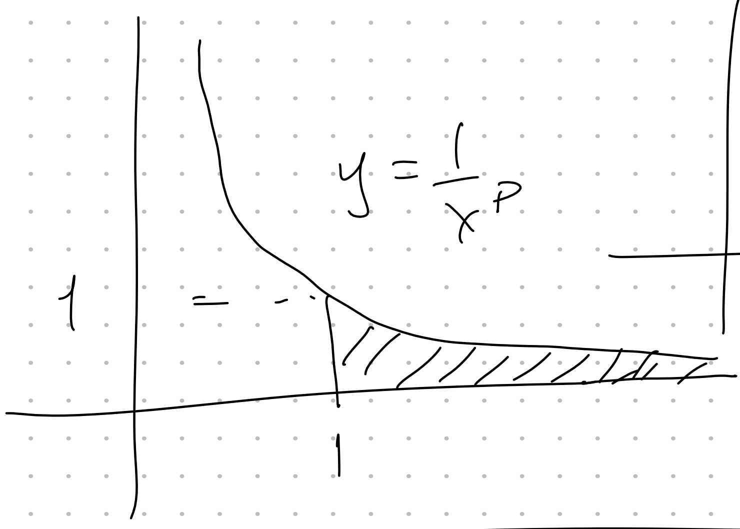
If $-p+1>0$: $t^{(-p+1)} \to \infty$ as $t\to\infty$

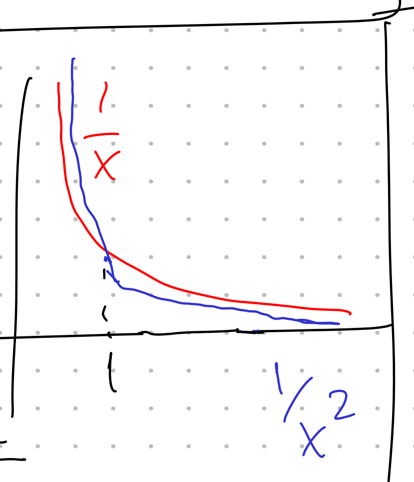
If $-p+1<0$: $t^{-p+1} = \dfrac{1}{t^{p-1}} \to 0$, $t\to\infty$

$-p+1>0 \iff 1>p$ DIV

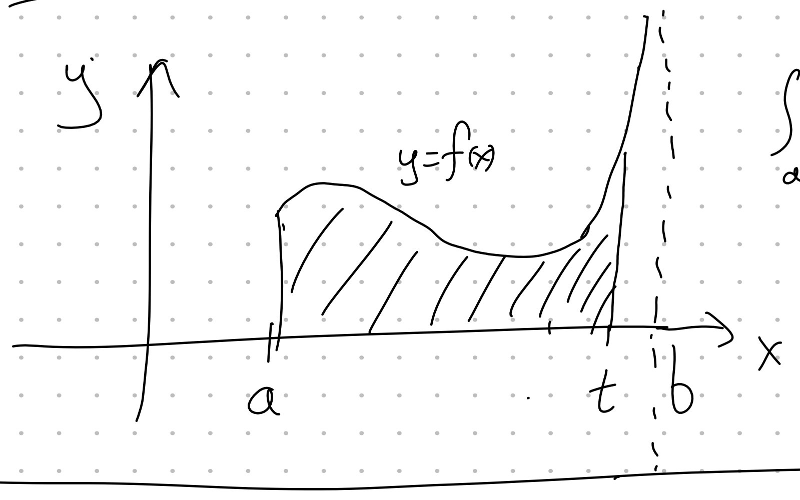
$-p+1<0 \iff 1<p$ CONV

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{CONV} & \text{if } p>1 \\ \text{DIV} & \text{if } p\le 1 \end{cases}}$$

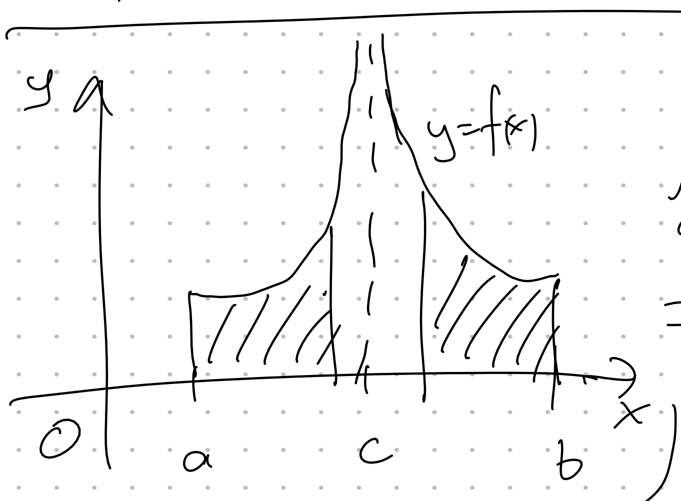




Type 2: Discontinuous Integrands



$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

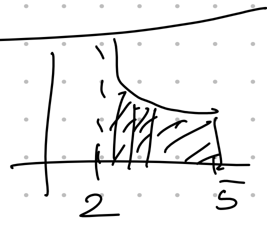


$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ so}$$

$$= \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$

Provided both limits exist.

Example: Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$



at $x=2$ we have a vertical asymptote

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx \qquad u=x-2, \quad du=dx$$

$$\int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{x-2} + C$$

$$= \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5 = \lim_{t\to 2^+} \left(2\sqrt{3} - 2\underbrace{\sqrt{t-2}}_{0}\right)$$

$$= 2\sqrt{3}$$

Example $\displaystyle\int_0^{\pi/2} \sec x\,dx$

Note that $\sec x = \dfrac{1}{\cos x}$ has a vert. asymp at $\pi/2$, so

$$\int_0^{\pi/2} \sec x\,dx = \lim_{t\to \pi/2^-} \int_0^t \sec x\,dx$$

$$= \lim_{t\to \frac{\pi}{2}^-} \ln|\sec x + \tan x|\Big|_0^t$$

$$= \lim_{t\to \frac{\pi}{2}^-} \left(\ln|\overset{+\infty}{\sec t} + \overset{+\infty}{\tan t}| - \ln|\sec 0 + \tan 0| \right)$$

DIV

Example: Find $\displaystyle\int_0^3 \frac{dx}{x-1}$

Vert. asymp. at $x=1$

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

$$= \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} + \lim_{t\to 1^+} \int_t^3 \frac{dx}{x-1}$$

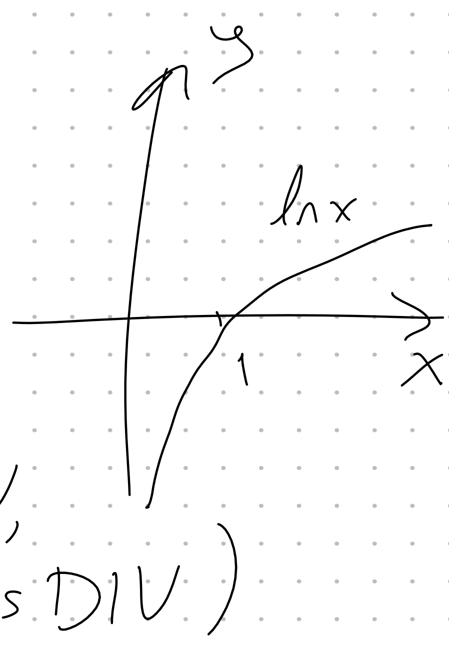
$$\int_0^t \frac{dx}{x-1} = \ln|x-1|\Big|_0^t = \ln|t-1| - \ln|0-1|$$

$$= \ln|t-1| - \underbrace{\ln 1}_{0}$$

$$\lim_{t\to 1^-} \ln|t-1| = -\infty$$

DIV

(If at least one is DIV, the whole thing is DIV)



Example $\displaystyle\int_0^1 \ln x\,dx$

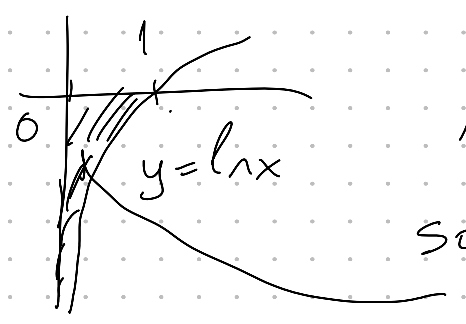
$$= \lim_{t\to 0^+} \int_t^1 \ln x\,dx$$

$u=\ln x \quad dv=dx$

$du=\frac{1}{x}dx \quad v=x$

$$\int \ln x\,dx = uv - \int v\,du = x\ln x - \int x\cdot\frac{1}{x}\,dx$$
$$= x\ln x - \int dx = x\ln x - x + C$$
$$\int_t^1 \ln x\,dx = x\ln x - x\Big|_t^1$$
$$= -1 - (t\ln t - t).$$

So
$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} -1 + t - t\ln t$$
$$= -1 + \lim_{t\to 0^+} t(1-\ln t) = -1 + \lim_{t\to 0^+}\frac{1-\ln t}{t^{-1}}$$
$$\overset{L'H}{=} -1 + \lim_{t\to 0^+}\frac{+\frac{1}{t}}{+t^{-2}} = -1 + \lim_{t\to 0^+} t^{\cancel{2}}\cdot\frac{1}{\cancel{t}}$$
$$= -1 + 0 = -1$$



$$\int_0^1 \ln x\,dx = -1$$

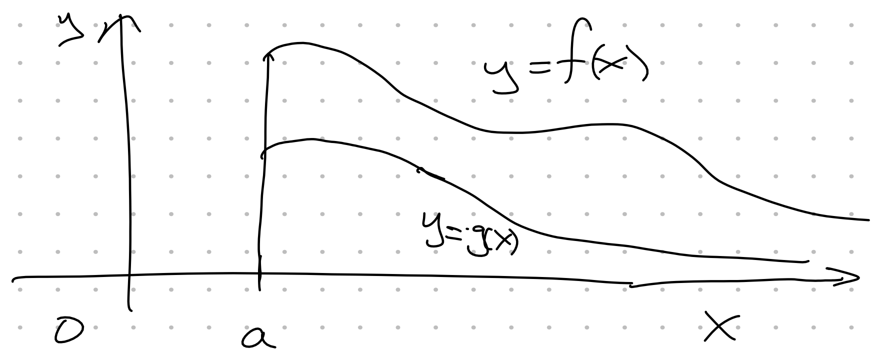
so Area = 1

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

a) If $\int_a^\infty f(x)\,dx$ is CONV, then $\int_a^\infty g(x)\,dx$ is CONV

b) If $\int_a^\infty g(x)\,dx$ is DIV, then $\int_a^\infty f(x)\,dx$ is DIV



Example Is $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ Conv. or DIV.?

Note that $\frac{1+e^{-x}}{x} > \frac{1}{x}$ since e^A is always positive.

Since $\int_1^\infty \frac{1}{x}\,dx$ is DIV (by p-test)

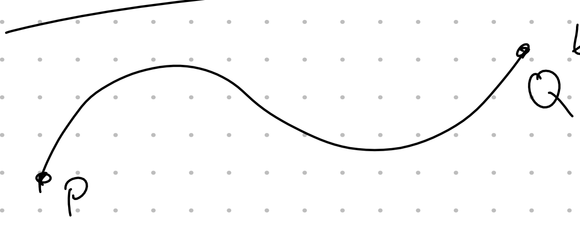
$\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is DIV by Comparison Theorem.

Example Show $\int_1^\infty \frac{1}{(x+1)^2}\,dx$ is Conv.

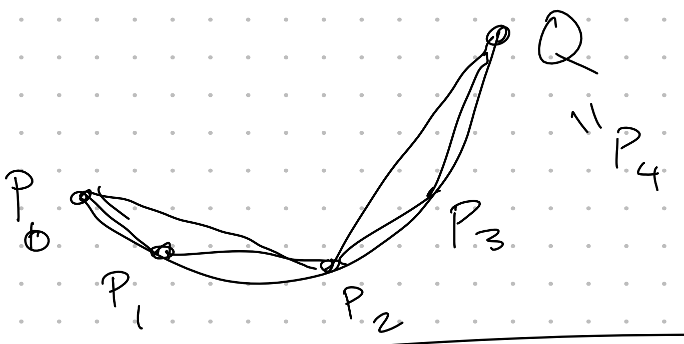
$$x+1 > x \geq 1$$
$$(x+1)^2 > x^2$$
$$0 \leq \frac{1}{(x+1)^2} < \frac{1}{x^2}$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ is CONV, $\int_1^\infty \frac{1}{(x+1)^2}\,dx$ is also CONV by Comparison Thm.

8.1 Arc Length



what is the length of this curve from P to Q



L = Total Length $\simeq \sum_{i=1}^{N} |P_{i-1}P_i|$ ← length of the line segment from P_{i-1} to P_i

In the limit

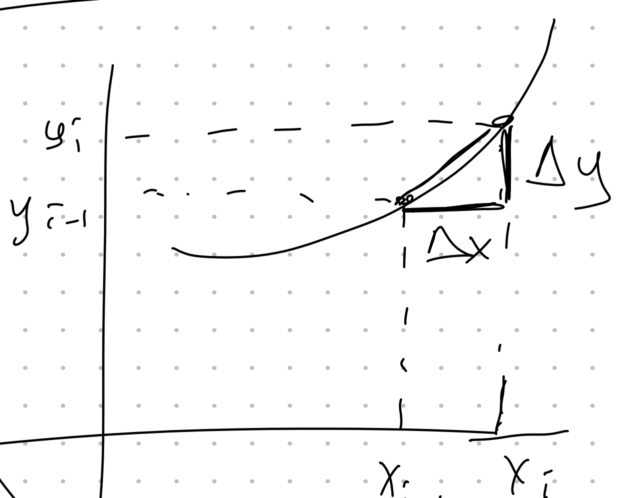
$$L = \lim_{N\to\infty}\sum_{i=1}^{N}|P_{i-1}P_i|$$

$|P_{i-1}P_i| = ?$ Say $P_i = (x_i, y_i)$

So $|P_{i-1}P_i| = \sqrt{(x_{i-1}-x_i)^2 + (y_{i-1}-y_i)^2} = \sqrt{\Delta x^2 + \Delta y^2}$

let $\Delta x = x_{i-1} - x_i$

$\Delta y = y_{i-1} - y_i$



$$\frac{\Delta y}{\Delta x} \simeq f'(x_i)$$

So $\Delta y \simeq f'(x_i)\,\Delta x$

So
$$|P_{i-1}P_i| = \sqrt{\Delta x^2 + \Delta y^2}$$
$$= \sqrt{\Delta x^2 + (f'(x_i)\Delta x)^2}$$
$$= \sqrt{\Delta x^2(1 + (f'(x_i))^2)}$$
$$= \sqrt{1 + (f'(x_i))^2}\,\Delta x$$

$$L = \lim_{N\to\infty}\sum_{i=1}^{N}\sqrt{1+f'(x_i)^2}\,\Delta x$$

$$\text{Arc Length} = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$