7.5 Strategy for Integration

1) Simplify

e.g. $\int \sqrt{x}(1+\sqrt{x})\,dx$

$= \int (\sqrt{x} + x)\,dx = \int (x^{1/2} + x)\,dx$

$= \frac{x^{3/2}}{3/2} + \frac{x^2}{2} + C = \frac{2}{3}x^{3/2} + \frac{x^2}{2} + C$

$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\,d\theta$

$= \int \cos^{\cancel{2}}\theta \frac{\sin\theta}{\cancel{\cos\theta}}\,d\theta = \int \cos\theta\sin\theta\,d\theta$

$u = \sin\theta \quad du = \cos\theta\,d\theta$

$= \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\sin^2\theta$

$\int (\sin x + \cos x)^2\,dx$

$= \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$

$= \int (1 + 2\sin x\cos x)\,dx$

$= \int (1 + \sin 2x)\,dx$ $\quad \sin 2x = 2\sin x\cos x$

$= x - \frac{\cos 2x}{2} + C$

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2) look for an obvious substitution

$\int \frac{3}{3}(x^2 + \frac{1}{3})\sqrt{x^3+x}\,dx$

$= \int \frac{1}{3}(3x^2+1)\sqrt{x^3+x}\,dx$

$u = x^3 + x \quad du = (3x^2+1)\,dx$

$= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{2}{9}(x^3+x)^{3/2} + C$

$\int \tan^{-1}(x)\,dx \quad u = \tan^{-1}(x) \quad dv = dx$

$du = \frac{1}{1+x^2}\,dx \quad v = x$

$= uv - \int v\,du$

$= x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx \quad u = 1+x^2 \quad du = 2x\,dx$

$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}\,du$

$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$

$= x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$

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3) Classify the integrand

- trignometric function $\sin, \cos$
- Rational function $\frac{P(x)}{Q(x)}$
- Integration by parts $u = \quad dv =$; $du = \quad v =$
- Radicals $\sqrt{\pm x^2 \pm a^2}$ inverse trig substitution

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$\int e^{\sqrt{x}}\,dx \quad u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$2\sqrt{x}\,du = dx$

$\int 2e^u\sqrt{x}\,du$

$= 2\int e^u u\,du$

Integration by parts:

$f = u \quad dg = e^u\,du$

$df = du \quad g = e^u$

$= 2\left(fg - \int g\,df\right)$

$= 2\left(ue^u - \int e^u\,du\right) = 2(e^u u - e^u + C)$

$= 2(e^{\sqrt{x}}\sqrt{x} - e^{\sqrt{x}} + C)$

$\int \frac{-13e^x - 72}{e^{2x} + 13e^x + 36}\,dx \quad u = e^x \quad du = e^x\,dx$

$\frac{du}{u} = \frac{du}{e^x} = dx$

$\int \frac{-13u - 72}{u^2 + 13u + 36}\,\frac{du}{u}$

$= \int \frac{-13u - 72}{u(u^2 + 13u + 36)}\,du$

$\frac{-13u-72}{u(u+4)(u+9)} = \frac{A}{u} + \frac{B}{u+4} + \frac{C}{u+9}$

$-13u - 72 = A(u+4)(u+9) + Bu(u+9) + Cu(u+4)$

$u = 0 \Rightarrow -72 = 36A \Rightarrow A = -2$

$u = -4 \Rightarrow -20 = -20B \Rightarrow B = 1$

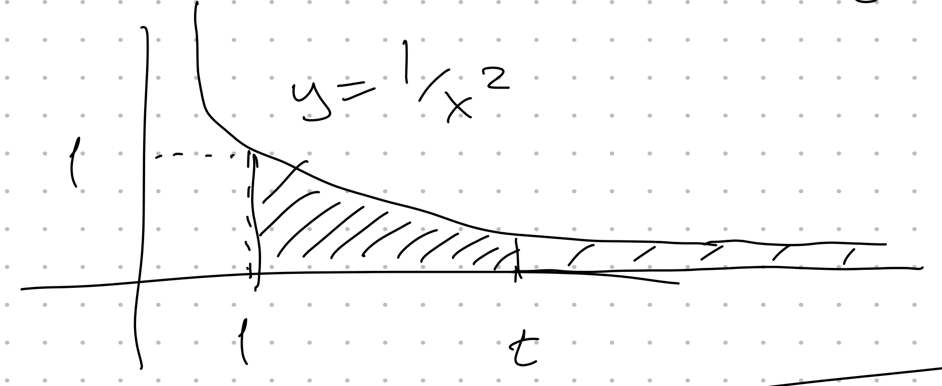
$u = -9 \Rightarrow 45 = 45C \Rightarrow C = 1$

So $\int \frac{-13u-72}{u(u+4)(u+9)}\,du = -2\int \frac{1}{u}\,du + \int \frac{1}{u+4}\,du + \int \frac{1}{u+9}\,du = -2\ln|u| + \ln|u+4| + \ln|u+9| + C$

$= -2\ln|e^x| + \ln|e^x+4| + \ln|e^x+9| + C$

7.8 Improper Integrals

These are the integrals where we have to deal with infinity.



In this case we write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1$$

$A(t)$: area under $\frac{1}{x^2}$ from 1 to t.

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = -x^{-1}\Big|_1^t$$

$$= -\left(\frac{1}{t} - \frac{1}{1}\right) = 1 - \frac{1}{t}$$

so now we see that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

Definition of an Improper Integral of Type 1

a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$ then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number.)

b) Similarly,

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

if the limit exists.

We call the improper integrals $\int_a^\infty f(x)dx$, and $\int_{-\infty}^a f(x)dx$ convergent if the corresponding limits exist, and divergent if they do not.

c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

Example 1 Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t$$

$$= \lim_{t\to\infty}(\ln t - \ln 1) = \infty$$

So $\int_1^\infty \frac{1}{x}\,dx$ is DIV.

Example Evaluate $\int_{-\infty}^0 xe^x\,dx$

$$= \lim_{t\to-\infty}\int_t^0 xe^x\,dx \quad \Big| \quad \int xe^x\,dx$$

$$\int xe^x\,dx \qquad u = x \quad dv = e^x dx$$
$$du = dx \quad v = e^x$$

$$= xe^x - \int e^x dx = \underline{xe^x - e^x} + C$$

$$= \lim_{t\to-\infty}\left(xe^x - e^x\right)\Big|_t^0$$

$$= \lim_{t\to-\infty}\left(0 - 1 - \left(te^t - e^t\right)\right)$$

$$= \lim_{t\to-\infty}\left(-1 + e^t(1-t)\right) \qquad 0\cdot\infty$$

$$= -1 + \lim_{t\to-\infty}\left(\frac{1-t}{e^{-t}}\right) \qquad \frac{\infty}{\infty}$$

$$\overset{L'H}{=} -1 + \lim_{t\to-\infty}\frac{-1}{-e^{-t}} = -1 + 0 = -1$$

$e^{-t} \to \infty$

So $\int_{-\infty}^0 xe^x\,dx = -1$ and it is CONV.

Example Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

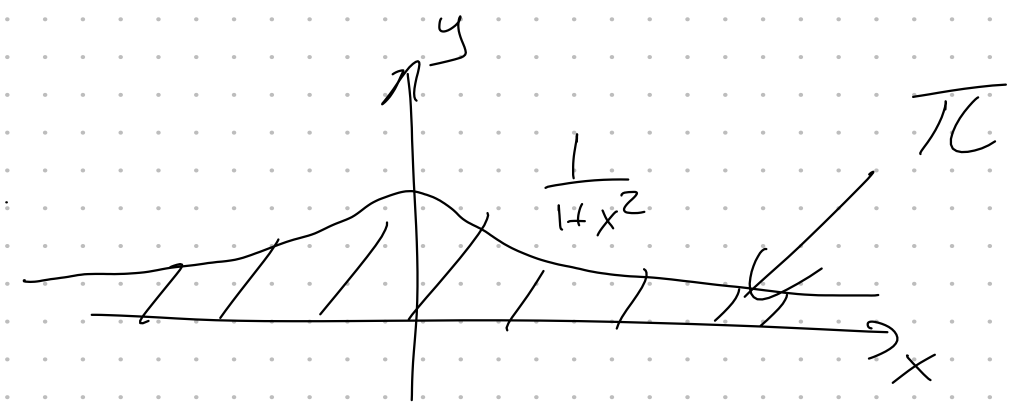
$$= \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

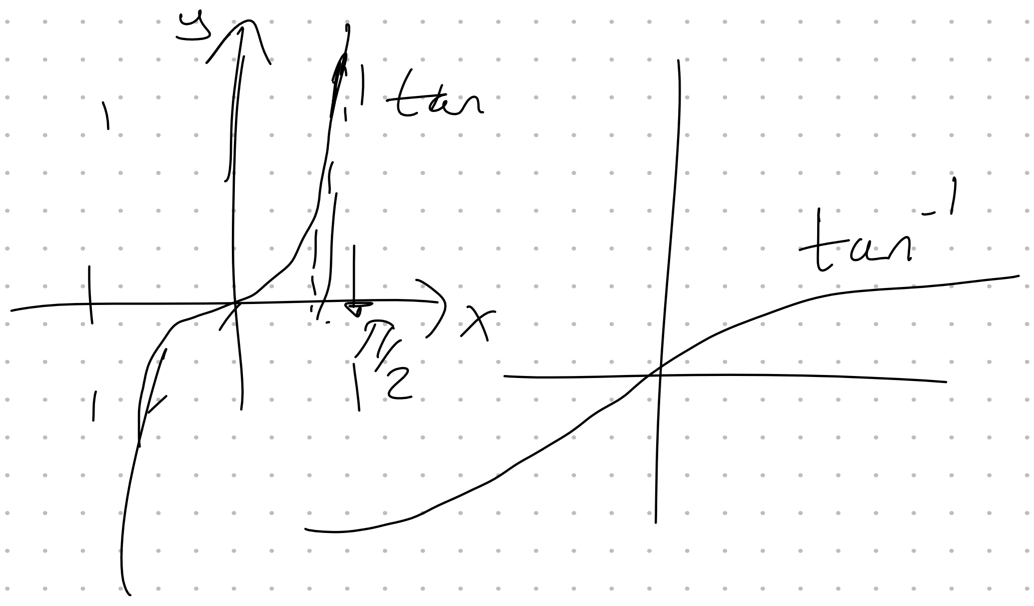
$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \tan^{-1}(x)\Big|_0^t = \lim_{t\to\infty}\left(\tan^{-1}(t) - \tan^{-1}(0)\right) = \frac{\pi}{2} - 0 = \pi/2$$

similarly, $\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \pi/2$ so $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

CONV.


y
$\frac{1}{1+x^2}$
π
x


y
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