Last time Q) How to integrate a rational function $\frac{P(x)}{Q(x)}$ = ?

First, we use long division

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \quad \text{so } \deg(R) < \deg(Q)$$

Now we only need to worry about $\frac{R(x)}{Q(x)}$. To handle $\frac{R(x)}{Q(x)}$ we factorize $Q(x)$.

Case I $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$

where a_ix+b_i is not a constant multiple of a_jx+b_j for $i \neq j$.

Then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Example Find

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx \qquad Q(x) = 2x^3+3x^2-2x$$

$$Q(x) = x(2x^2+3x-2)$$

$$\begin{matrix} 2x & & -1 \\ x & & 2 \end{matrix}$$

$$= x(2x-1)(x+2)$$

So $\frac{R(x)}{Q(x)} = \boxed{\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}}$

Now we multiply both sides by $Q(x) = x(2x-1)(x+2)$

$$\frac{x^2+2x-1}{\cancel{Q(x)}}\cancel{Q(x)} = \frac{A}{\cancel{x}}\cancel{x}(2x-1)(x+2) + \frac{B}{\cancel{2x-1}}x\cancel{(2x-1)}(x+2) + \frac{C}{\cancel{x+2}}x(2x-1)\cancel{(x+2)}$$

$$(*) \quad x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

plugin $x=0$ in $(*)$ $\quad -1 = A(-1)(2) + \cancel{B0} + \cancel{C0} = -2A \implies A = 1/2$

$x = \frac{1}{2}$ in $(*)$ $\quad \frac{1}{4} \cancel{+1} \cancel{-1} = A0 + B\frac{1}{2}\left(\frac{5}{2}\right) + C0 = \frac{5B}{4}$

$$\frac{1}{\cancel{4}} = \frac{5B}{\cancel{4}} \qquad B = \frac{1}{5}$$

$x = -2$ in $(*)$ $\quad \cancel{4} \cancel{-4} -1 = A0 + B0 + C(-2)(-5)$

$$-1 = +10C \qquad C = \frac{-1}{10}$$

$$\int \frac{R(x)}{Q(x)}\,dx = \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$

$u = 2x-1 \quad du = 2dx$

$$\int \frac{1}{2x-1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x-1| + C$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\frac{\ln|2x-1|}{2} - \frac{1}{10}\ln|x+2| + C$$

Example Find $\int \frac{1}{x^2-4}\,dx$

$$Q(x) = x^2-4 = (x-2)(x+2)$$

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

Multiply by $Q(x)$

$$1 = A(x+2) + B(x-2)$$

$$1 = Ax + 2A + Bx - 2B$$

$$1 = \underbrace{(A+B)x}_{x\text{-term}} + \underbrace{(2A-2B)}_{\text{constant term}}$$

$$1 = 0x + 1 = (A+B)x + (2A-2B)$$

$$0 = A+B$$

$$1 = 2A - 2B$$

$A = -B \longrightarrow 1 = 2(-B) - 2B = -4B$

$$B = \frac{-1}{4} \qquad A = -B = \frac{1}{4}$$

So $\int \frac{1}{x^2-4}\,dx = \int \left(\frac{1/4}{x-2} + \frac{-1/4}{x+2}\right)dx$

$$= \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

Case II $Q(x)$ is a product of linear factors, some of which are repeated.

$$\frac{R(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_r}{(ax+b)^r}$$

Example

Find $\int \frac{4x}{x^3-x^2-x+1}\,dx$

$$Q(x) = x^3 - x^2 - x + 1$$

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Note that $Q(1) = 0$ so $(x-1)$ is a factor of $Q(x)$.

$Q(x) = (x-1)(ax^2+bx+c) = ax^3 + \_\_\_\_\_ - c = x^3 - x^2 - x + 1$

So we must have $a = 1$, $c = -1$. Thus, $x^3 - x^2 - x + 1 = (x-1)(x^2 + bx - 1)$

$= x^3 - x^2 - x + 1 + \underbrace{b}_{0}x(x-1) = x^3 - x^2 - x + 1$

$Q(x) = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$

$\dfrac{4x}{(x-1)^2(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$ multiply both sides by $Q(x)$.

(*) $4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$

$x=1$ in (*) gives us $4 = 0 + 0 + 2C$

$C = 2$

$x=-1$ in (*) $-4 = 4A + 0 + 0$

$A = -1$

$x = 0$ in (*) gives us

$0 = A - B + C = -1 - B + 2$

$B = 1$

$\displaystyle\int \frac{4x}{Q(x)}dx = -\int \frac{1}{x+1}dx + \int \frac{1}{x-1}dx + 2\int \frac{1}{(x-1)^2}dx = -\ln|x+1| + \ln|x-1| + 2\int \frac{1}{(x-1)^2}dx$

$\displaystyle\int \frac{1}{(x-1)^2}dx = \int (x-1)^{-2}dx$ $u = x-1$ $du = dx$ $\displaystyle = \int u^{-2}du = -u^{-1} + C = -(x-1)^{-1} + C = -\frac{1}{x-1} + C$

Case III $Q(x)$ contains irreducible quadratic factors none of which is repeated.

irreducible means there are no real solutions

e.g. x^2+4 or $b^2 - 4ac < 0$

$\dfrac{R(x)}{ax^2+bx+c} = \dfrac{Ax+B}{ax^2+bx+c}$

e.g. $\dfrac{x}{(x-2)(x^2+1)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{x^2+4}$

$\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Example $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}dx$

$Q(x) = x^3 + 4x = x(x^2+4)$

$\dfrac{2x^2 - x + 4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$

$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$

$x = 0 \Rightarrow 4 = 4A + 0$

$A = 1$

$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$

$= (A+B)x^2 + Cx + 4A$

$2 = A + B = B + 1 \Rightarrow B = 1$

$-1 = C$

$\displaystyle\int \frac{R(x)}{Q(x)}dx = \int \frac{1}{x}dx + \int \frac{x-1}{x^2+4}dx$

$\displaystyle = \ln|x| + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$

$u = x^2+4$

$du = 2x\,dx$

$\displaystyle = \ln|x| + \frac{1}{2}\int \frac{1}{u}du - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

Case IV $Q(x)$ contains a repeated irreducible quadratic factor.

$\dfrac{R(x)}{(ax^2+bx+c)^r} = \dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$

Example Write out the form of the partial fraction decomposition of the function

$\dfrac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+x+1} + \dfrac{Ex+F}{x^2+1} + \dfrac{Gx+H}{(x^2+1)^2} + \dfrac{Ix+J}{(x^2+1)^3}$