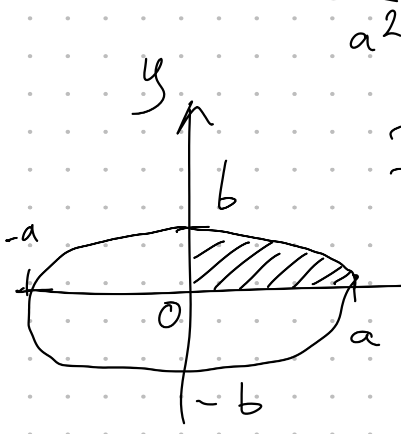
First Midterm: Tuesday 2/25, 8–9:15 am in DEWEY 1-101

Example Find the area of the ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \qquad y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$



It is enough to find the area of the shaded region and multiply it by 4

$$y=+\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}=b\sqrt{1-\frac{x^2}{a^2}}$$

$$A=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=4\int_0^a b\sqrt{\frac{1}{a^2}(a^2-x^2)}\,dx$$

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$x=a\sin t \qquad dx=a\cos t\,dt$

$x=0=a\sin t \qquad t=0$

$x=a=a\sin t \qquad t=\pi/2$

$$\rightarrow =\frac{4b}{\not a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2 t}\;a\cos t\,dt$$

$$=4b\int_0^{\pi/2}\sqrt{a^2(1-\sin^2 t)}\,\cos t\,dt$$

$$=4b\int_0^{\pi/2} a\sqrt{\cos^2 t}\,\cos t\,dt$$

$(\cos^2 t+\sin^2 t=1)\rightarrow(1-\sin^2 t=\cos^2 t)$

$$=4ab\int_0^{\pi/2}\cos^2 t\,dt=4ab\,\frac{1}{2}\int_0^{\pi/2}(1+\cos 2t)\,dt$$

$\cos^2 t=\frac{1}{2}(1+\cos 2t)$

$$=2ab\left[t+\frac{\sin 2t}{2}\right]\Big|_0^{\pi/2}$$

$$=2ab\left(\frac{\pi}{2}+0-(0)\right)=\pi ab$$

Note if $a=b=r$ (so we have a circle of radius r), we get

$A=\pi ab=\pi r^2$

Example Find

$$I=\int\frac{1}{x^2\sqrt{x^2+4}}\,dx$$

$\sin^2 t+\cos^2 t=1$

$\sec^2 t=1+\tan^2 t$

$x=2\tan t \qquad dx=2\sec^2 t\,dt$

$-\frac{\pi}{2}<t<\frac{\pi}{2}$

$$I=\int\frac{2\sec^2 t}{4\tan^2 t\sqrt{4\tan^2 t+4}}\,dt$$

$$=\int\frac{\not 2\sec^2 t}{{}_2\not 4\tan^2 t\sqrt{4(\tan^2 t+1)}}\,dt$$

$$=\frac{1}{4}\int\frac{\sec^2 t}{\tan^2 t\sqrt{\sec^2 t}}\,dt \qquad \boxed{\sqrt{\sec^2 t}=|\sec t|}$$

$$=\frac{1}{4}\int\frac{\sec^{\not 2} t}{\tan^2 t\,\not{\sec t}}\,dt \qquad \sec x=\frac{1}{\cos x} \qquad \tan x=\frac{\sin x}{\cos x}$$

$$=\frac{1}{4}\int\frac{\frac{1}{\cos t}}{\frac{\sin^2 t}{\cos^2 t}}\,dt=\frac{1}{4}\int\frac{\cos^{\not 2} t}{\sin^2 t}\,\frac{1}{\not{\cos t}}\,dt$$

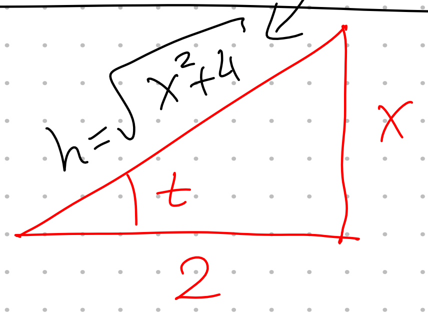
$$=\frac{1}{4}\int\frac{\cos t}{\sin^2 t}\,dt \qquad u=\sin t \qquad du=\cos t\,dt$$

$$=\frac{1}{4}\int\frac{du}{u^2}=\frac{1}{4}\int u^{-2}\,du=\frac{-1}{4}u^{-1}+C$$

$$=-\frac{1}{4\sin t}+C$$

$x=2\tan t$

so $\tan t=\frac{x}{2}$



$h^2=x^2+2^2$

$h^2=x^2+4$

$h=\sqrt{x^2+4}$

So $\sin t=\frac{x}{\sqrt{x^2+4}}$

$$\boxed{=-\frac{1}{4}\frac{\sqrt{x^2+4}}{x}+C}$$

Example Find $\int\frac{x}{\sqrt{x^2+4}}\,dx=I$

Here an inverse trig sub would work but a simple u sub is much faster!

$u=x^2+4 \qquad du=2x\,dx$

$$I=\frac{1}{2}\int\frac{du}{\sqrt{u}}=\frac{1}{2}\int u^{-1/2}\,du$$

$$=\frac{1}{\not 2}\not 2u^{1/2}+C=\sqrt{x^2+4}+C$$

Example $\int\frac{1}{\sqrt{1+(7x-8)^2}}\,dx=I$

$u=7x-8 \qquad du=7\,dx$

$$I=\frac{1}{7}\int\frac{1}{\sqrt{1+u^2}}\,du \qquad u=\tan t\ \left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right) \qquad du=\sec^2 t\,dt$$

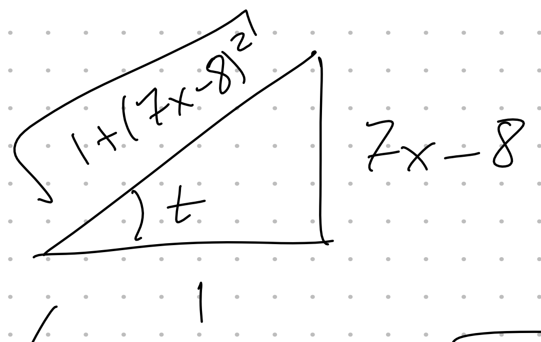
$$=\frac{1}{7}\int\frac{\sec^2 t\,dt}{\sqrt{1+\tan^2 t}}=\frac{1}{7}\int\frac{\sec^2 t\,dt}{\sqrt{\sec^2 t}}$$

$$=\frac{1}{7}\int\frac{\sec^{\not 2} t\,dt}{\not{\sec t}}$$

$$=\frac{1}{7}\int\sec t\,dt=\frac{1}{7}\ln|\sec t+\tan t|+C$$

Note that

$\tan t=u=\frac{7x-8}{1}$



$\hookrightarrow \sec t = \sqrt{1+(7x-8)^2}$

$\tan t = 7x-8$

$$I = \frac{1}{7}\ln\left|\sqrt{1+(7x-8)^2} + 7x-8\right| + C$$

7.4 Integration by Partial Fractions

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{du}{u} \qquad u = ax+b, \quad du = a\,dx$$

$$= \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$$

So when we start with a rational function $f(x) = \frac{P(x)}{Q(x)}$ ← Polynomials by definition, we might be able to express it as a sum of simpler terms (called partial fractions) and integrate them.

So for example,

$$\int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx = 2\ln|x-1| - \ln|x+2| + C$$

$$\int \frac{2(x+2)-(x-1)}{(x-1)(x+2)}\,dx$$

$$= \int \frac{x+5}{x^2+x-2}\,dx \qquad (P(x) = x+5,\ Q(x) = x^2+x-2)$$

If $\deg(P(x)) \geq \deg(Q(x))$ then we first have to use polynomial long division to simplify $\frac{P(x)}{Q(x)}$ as

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Example Find $\int \frac{x^3+x}{x-1}\,dx$

since $\deg(x^3+x) = 3 \geq \deg(x-1) = 1$, we first use long division.

$$\begin{array}{r} x^2 + x + 2 \\ x-1 \overline{)\, x^3 \qquad + x} \\ -(x^3 - x^2) \downarrow \\ \hline x^2 + x \\ -(x^2 - x) \\ \hline 2x \\ -(2x-2) \\ \hline +2 \end{array}$$

$\frac{x^3}{x} = x^2$

$\frac{x^2}{x} = x$

$\frac{2x}{x} = 2$

So

$$x^3+x = (x-1)(x^2+x+2) + 2$$

thus

$$\frac{x^3+x}{x-1} = \frac{(x-1)(x^2+x+2)}{x-1} + \frac{2}{x-1}$$

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2+x+2 + \frac{2}{x-1}\right)dx$$

$$\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

So from now on, we can focus on integrating $\frac{R(x)}{Q(x)}$ for $\deg(R(x)) < \deg(Q(x))$

We will learn how to express $\frac{R(x)}{Q(x)}$ as sums of partial fractions in a few different cases.

$$\text{Partial fractions} = \frac{A}{(Bx+C)^k}, \quad \frac{Ax+B}{(Cx^2+Dx+E)^k}$$

To get this decomposition we have to factor $Q(x)$ first.

Case I: $Q(x)$ only has linear and distinct factors.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

so that no two of a_ix+b_i have the same root!

that is if $a_ix+b_i = 0$

then $x = -\frac{b_i}{a_i}$ are all distinct.