Example Find $I=\int \sin^4 x\,dx$

Recall
$\sin^2 x = \frac{1-\cos 2x}{2}$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$

$$I=\int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$=\frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx$$

Using with $2x$ in place of x,

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$=\frac{1}{4}\left(x-\sin 2x+\frac{1}{2}\int (1+\cos 4x)\,dx\right)$$

$$=\frac{1}{4}\left(x-\sin 2x+\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right)+C$$

Next, we think about integrals of the form $\int \tan^m x \sec^n x\,dx$

Example Evaluate $\underbrace{\int \tan^6 x \sec^4 x\,dx}_{= I}$

$\sec^2 x = 1+\tan^2 x$

$u=\tan x$
$du=\sec^2 x\,dx$

$u=\sec x$
$du=\sec x\tan x\,dx$

$$I=\int \underbrace{\tan^6 x}_{u^6}\ \underbrace{\sec^2 x}_{=1+\tan^2 x = 1+u^2}\ \underbrace{\sec^2 x\,dx}_{du}$$

$$I=\int u^6(1+u^2)\,du=\int (u^6+u^8)\,du$$

$$=\frac{u^7}{7}+\frac{u^9}{9}+C=\frac{\tan^7 x}{7}+\frac{\tan^9 x}{9}+C$$

Example Find $I=\int \tan^5 t\sec^7 t\,dt$

$u=\sec t$ $\quad du=\sec t\tan t\,dt$

$$I=\int \underbrace{\tan^4 t}_{=(\tan^2 t)^2}\ \underbrace{\sec^6 t}_{u^6}\ \underbrace{\sec t\tan t\,dt}_{du}$$

$$I=\int (\tan^2 t)^2 u^6\,du$$

$(\tan^2 t)^2 = (\sec^2 t-1)^2 = (u^2-1)^2$

$$=\int (u^2-1)^2 u^6\,du$$

$$=\int (u^4-2u^2+1)u^6\,du$$

$$=\int (u^{10}-2u^8+u^6)\,du$$

$$=\frac{u^{11}}{11}-2\frac{u^9}{9}+\frac{u^7}{7}+C$$

$$=\frac{\sec^{11} t}{11}-2\frac{\sec^9 t}{9}+\frac{\sec^7 t}{7}+C$$

$\int \tan x\,dx \overset{①}{=} \ln|\sec x|+C$

$\int \sec x\,dx \overset{②}{=} \ln|\sec x+\tan x|+C$

because

$$\int \sec x\,dx=\int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}dx$$

$$=\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$$

$u=\sec x+\tan x$
$du=(\sec x\tan x+\sec^2 x)\,dx$

$$=\int \frac{1}{u}du=\ln|u|+C$$

$$=\ln|\sec x+\tan x|+C$$

Example Find $I=\int \tan^3 x\,dx$

$$I=\int \tan x\tan^2 x\,dx=\int \tan x(\sec^2 x-1)\,dx$$

$$=\int \tan x\sec^2 x\,dx-\int \tan x\,dx$$

$u=\tan x$
$du=\sec^2 x\,dx$

①

$$=\int u\,du-\ln|\sec x|+C$$

$$=\frac{u^2}{2}-\ln|\sec x|+C$$

$$=\frac{\tan^2 x}{2}-\ln|\sec x|+C$$

$$=\frac{\sec^2 x}{2}-\ln|\sec x|+C_1$$

this is the answer you would get if you used $u=\sec x$ sub. but they only differ by a constant so they are equivalent.

Example Find $\int \sec^3 x\,dx = I$

(u-sub does not give us a simple answer here), so we try integration by parts.

$u=\sec x$ $\quad dv=\sec^2 x\,dx$

$du=\sec x\tan x\,dx$ $\quad v=\tan x$

$$=\int u\,dv=uv-\int v\,du$$

$$=\tan x\sec x-\int \tan^2 x\sec x\,dx$$

$$=\tan x\sec x-\int (\sec^2 x-1)\sec x\,dx$$

$$=\tan x\sec x-\underbrace{\int \sec^3 x\,dx}_{I}+\underbrace{\int \sec x\,dx}_{=\ln|\sec x+\tan x|}$$

$$I=\tan x\sec x-I+\ln|\sec x+\tan x|+C$$

$$2I=\tan x\sec x+\ln|\sec x+\tan x|+C$$

$$I=\frac{1}{2}\left(\tan x\sec x+\ln|\sec x+\tan x|+C\right)$$

Recall

a) $\sin A\cos B=\frac{1}{2}\left(\sin(A-B)+\sin(A+B)\right)$

b) $\sin A\sin B=\frac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)$

c) $\cos A\cos B=\frac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)$

Example Find

$$\int \sin 4x \cos 5x \, dx$$

$$= \frac{1}{2}\int \left(\sin(-x) + \sin(9x)\right) dx$$

$$= \frac{1}{2}\left(\cos(-x) - \frac{\cos(9x)}{9}\right) + C$$

7.3 Trigonometric Substitution

If we had

$$\int f(g(t))\, g'(t)\, dt$$

we could use a substitution of the form $x = g(t)$ $dx = g'(t)\,dt$ to get

$$\int f(g(t))\, g'(t)\, dt = \int f(x)\, dx.$$

Sometimes when we start with $\int f(x)\,dx$ and make $x = g(t)$, $dx = g'(t)\,dt$

sub, $\int f(x)\,dx = \int f(g(t))\, g'(t)\,dt$

the second integral we get is actually is simpler. Such a sub. is called inverse substitution.

For example

$f(x)$	substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$	$1 - \sin^2 t = \cos^2 t$
	$x = a\cos t,\ 0 \le t \le \pi$	$1 - \cos^2 t = \sin^2 t$
$\sqrt{a^2 + x^2}$	$x = a\tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$	$1 + \tan^2 t = \sec^2 t$
$\sqrt{x^2 - a^2}$	$x = a\sec t,\ 0 \le t < \frac{\pi}{2}$	$\sec^2 t - 1 = \tan^2 t$

Example Find $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\, dx = I$

$x = 3\cos t \quad (0 \le t \le \pi)$

$dx = -3\sin t\, dt$

$$I = \int \frac{\sqrt{9 - 9\cos^2 t}}{9\cos^2 t}(-3\sin t\,dt)$$

$$= -\cancel{3}\int \frac{\sqrt{9(1 - \cos^2 t)}}{\cancel{9}_{3}\cos^2 t}\sin t\,dt$$

$$= -\frac{1}{\cancel{3}}\int \frac{\cancel{\sqrt{9}}\sqrt{\sin^2 t}}{\cos^2 t}\sin t\,dt$$

since $0 \le t \le \pi$, $\sin t \ge 0$

so $\sqrt{\sin^2 t} = |\sin t| = \sin t$.

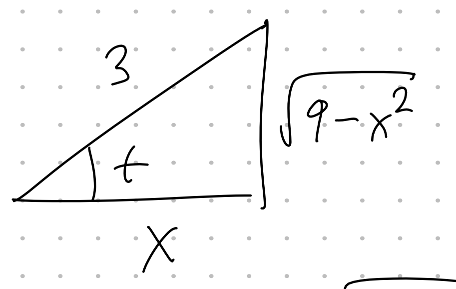
$$I = -\int \frac{\sin^2 t}{\cos^2 t}\,dt$$

$$= -\int \tan^2 t\,dt = -\int (\sec^2 t - 1)\,dt$$

$$= -\int \sec^2 t\,dt + \int 1\,dt$$

$$= -\tan t + t + C$$

initial subs.



$x = 3\cos t$

$\cos t = \frac{x}{3}$

$\Downarrow$

$t = \cos^{-1}\left(\frac{x}{3}\right)$

$\Rightarrow \tan t = \dfrac{\sqrt{9 - x^2}}{x}$

$$= -\frac{\sqrt{9 - x^2}}{x} + \cos^{-1}\left(\frac{x}{3}\right) + C$$