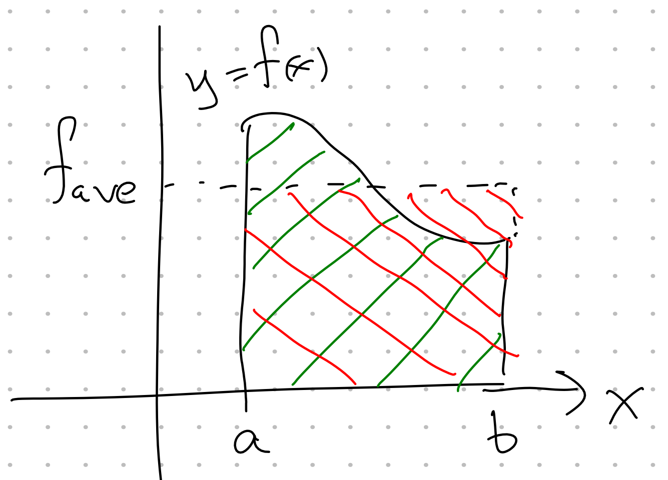
6.5 Average Value of a function



Area = Area gives us

$$f_{ave}(b-a) = \int_a^b f(x)\,dx$$

So $$f_{ave} = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

Example Find the average value of $f(x) = 1+x^2$ on the interval $[-1,2]$

$$f_{ave} = \frac{1}{2-(-1)} \cdot \int_{-1}^{2} (1+x^2)\,dx$$

$$= \frac{1}{3}\left(x+\frac{x^3}{3}\right)\Big|_{-1}^{2} = \frac{1}{3}\left(2+\frac{8}{3}\right) - \frac{1}{3}\left(-1-\frac{1}{3}\right)$$

$$= 2$$

(Similar question based on the previous example:) $f(x) = 1+x$ has an average value of 2 over an interval $[-1, b]$. find the possible values of b.

In this case we would have

$$2 = \frac{1}{b-(-1)} \int_{-1}^{b} (1+x)\,dx$$

then integrate the right hand side and solve for b!

Thm (The Mean Value Theorem for integrals) If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$

Example Verify the MVT for $f(x) = 1+x^2$ on $[-1,2]$.

From the above example, we know that $f_{ave} = 2$. So we are only looking for $-1 \le c \le 2$ such that $f(c) = 2$.

$$1+c^2 = 2 \rightarrow c^2 = 1$$

then $c = \pm 1$. Note that both $+1$ and -1 are in $[-1,2]$.

Chp 7 Techniques of Integration

7.1 Integration by parts

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx}(x)\,g(x) + f(x)\,\frac{dg}{dx}(x)$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$f(x)g'(x) = (f(x)g(x))' - f'(x)g(x)$$

$$\int f(x)g'(x)\,dx = \int (f(x)g(x))'\,dx - \int f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

let $u = f(x)$ and $v = g(x)$

$du = f'(x)\,dx$ $\quad dv = g'(x)\,dx$

So the last equation becomes

$$\int u\,dv = uv - \int v\,du$$

Example Find $\int x\sin x\,dx$

$u = x$ $\quad dv = \sin x\,dx$

$du = dx$ $\quad v = -\cos x$. so

$$\int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Example Find $\int \ln x\,dx$

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad v = x$

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int \frac{\cancel{x}}{\cancel{x}}\,dx$$

$$= x\ln x - x + C$$

Example Find $\int t^2 e^t\,dt$

$u = t^2$ $\quad dv = e^t dt$

$du = 2t\,dt$ $\quad v = e^t$

$$\int u\,dv = t^2 e^t - 2\int t e^t\,dt$$

We use integration by parts again to compute $\int t e^t\,dt$

$\int t e^t\,dt$ $\quad u = t$ $\quad dv = e^t dt$

$\quad du = dt$ $\quad v = e^t$

$$\| $$

$$t e^t - \int e^t\,dt = t e^t - e^t + C$$

So putting it all together we get

$$\int t^2 e^t\,dt = t^2 e^t - 2\left(t e^t - e^t + C\right)$$

$$= t^2 e^t - 2te^t + 2e^t + C_1 \text{ where}$$

$C_1 = -2C$

Example Find $\int e^x \sin x\,dx$

First call this integral I. So,

$$I = \int e^x \sin x\,dx$$

$u = \sin x$ $\quad dv = e^x dx$

$du = \cos x\,dx$ $\quad v = e^x$

① $I = uv - \int v\,du = e^x \sin x - \underbrace{\int e^x \cos x\,dx}_{J}$

$J = \int e^x \cos x\,dx$

$u = \cos x \qquad dv = e^x\,dx$

$du = -\sin x\,dx \qquad v = e^x \qquad$ So

② $J = uv - \int v\,du = e^x \cos x + \underbrace{\int e^x \sin x\,dx}_{I}$

So the two equations we got are

①: $I = e^x \sin x - J$

②: $J = e^x \cos x + I$

$I = e^x \sin x - (e^x \cos x + I)$

$I = e^x \sin x - e^x \cos x - I$

$2I = e^x \sin x - e^x \cos x$

So $\int e^x \sin x\,dx = I = \frac{1}{2} e^x (\sin x - \cos x) + C$

Example Find

$\int \tan^{-1}(x)\,dx \qquad u = \tan^{-1}(x) \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx$

$u = 1 + x^2$

$du = 2x\,dx \qquad$ so $\qquad x\,dx = \frac{1}{2}\,du$

and $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|1+x^2| + C = \frac{1}{2}\ln(1+x^2) + C$

So

$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) - C$

7.2 Trigonometric Integrals

First we think about the integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$.

(We allow 0 or negative powers.)

If at least one of m or n is odd a simple u-sub and the identity $\cos^2 x + \sin^2 x = 1$ will be enough to solve the problem.

Example Find $\int \cos^3 x\,dx$

$u = \sin x \qquad du = \cos x\,dx$

$\int \cos^3 x\,dx = \int \underbrace{\cos^2 x}_{?}\,\underbrace{\cos x\,dx}_{du}$

$= \int \cos^2 x\,du = \int (1 - \sin^2 x)\,du$

$= \int (1-u^2)\,du = u - \frac{u^3}{3} + C$

$= \sin x - \frac{\sin^3 x}{3} + C$

Example Find $\int \sin^5 x \cos^2 x\,dx$

$= \int \sin^4 x \cos^2 x \sin x\,dx$

So if we make $u = \cos x$ sub.

$du = -\sin x\,dx$

$-\int \sin^4 x \cos^2 x\,du$

$= -\int (\sin^2 x)^2 u^2\,du \qquad \cos^2 x + \sin^2 x = 1$

$= -\int (1 - \cos^2 x)^2 u^2\,du$

$= -\int (1-u^2)^2 u^2\,du$

$= -\int (1 - 2u^2 + u^4) u^2\,du$

$= -\int (u^2 - 2u^4 + u^6)\,du$

$= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C$

$-\left(\frac{\cos^3 x}{3} - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x\right) + C$

What do we do if both m and n are even?

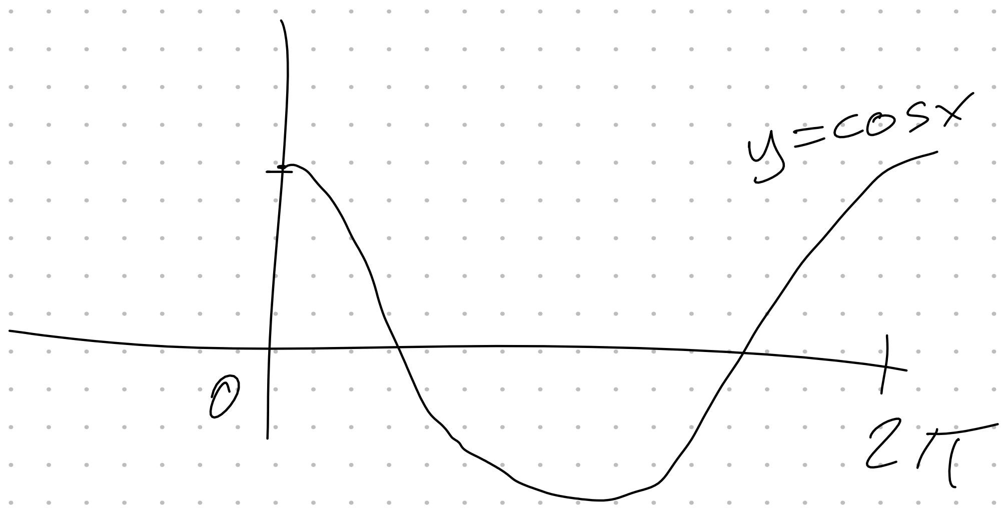
We use the double angle formulas:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Example $\int_0^\pi \sin^2 x\,dx = \frac{1}{2}\int_0^\pi (1 - \cos 2x)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^\pi = \frac{\pi}{2}$



$\int_0^\pi \cos(2x)\,dx = 0$