6.4 Work

Work = Force × Distance

$W = F d$ If force is constant!

SI metric system: Units: d: m (meters)

F: N (Newtons)

$F = ma$ $(kg\ m/s^2)$

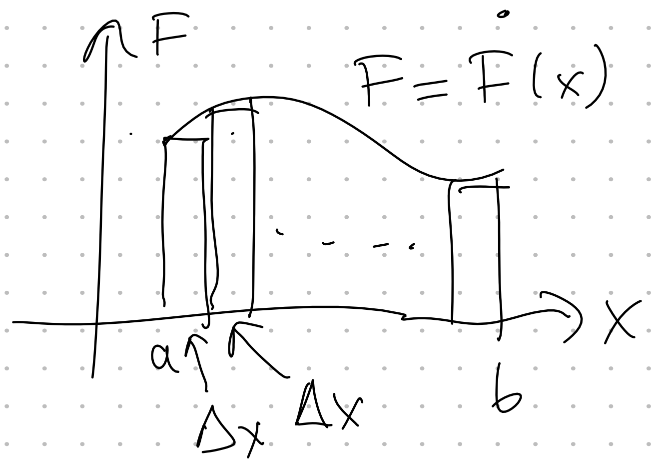
So $1N = 1\ kg\ m/s^2$

b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

(b) $F = 20$ $d = 6$

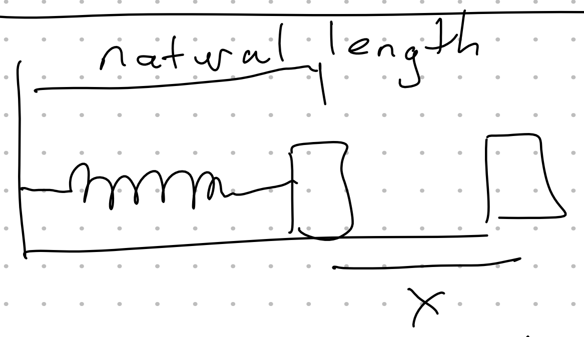
$W = 20(6) = 120$ ft-lb

What if the force applied is not constant?



The total Work is

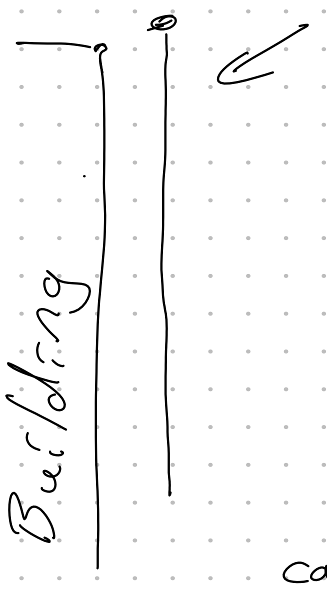
$$W = \int_a^b F(x)dx$$



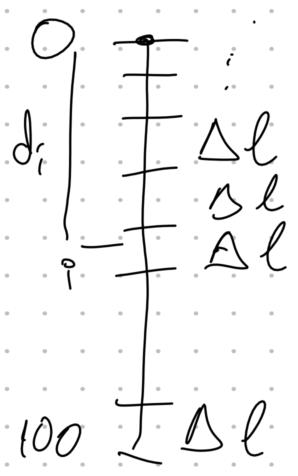
We pull the object x meters away from its natural equilibrium.

Then Hooke's Law say

$F = kx$ where k is the spring constant.



Example A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?



Note that the cable weights 2 pounds per foot (200lb/100ft)

So a piece of rope that is Δl ft long weights $2\Delta l = \Delta F$

$$W \approx \sum_{i=1}^{n} 2i\Delta l^2 \qquad \Delta l = \frac{100}{n}$$

$x_i = i\Delta l$

$$\approx \sum_{i=1}^{n} 2 x_i \Delta l \qquad (l = x)$$

So the units of work are N·m or $kg \frac{m^2}{s^2} = J$ (Joules)

US system:

Force lb (the pound)

dist. ft

Work ft-lb (foot-pound)

Examples

a) How much work is done in lifting a 1.2 kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that $g = 9.8\ m/s^2$

(a) $F = ma = mg = 1.2(9.8)\ kg\,m/s^2$

$d = 0.7\ m$

$W = 1.2(9.8)0.7\ J$

$\simeq 8.2\ J$

Example When a particle is located a distance x feet from the origin a force of x^2+2x pounds acts on it. How much work is done in moving it from $x=1$ to $x=3$?

$$W = \int_1^3 (x^2+2x)dx = \frac{x^3}{3} + x^2 \Big|_1^3$$

$$= \left(\frac{27}{3}+9\right) - \left(\frac{1}{3}+1\right) = \frac{50}{3} \text{ ft-lb}$$

Example A force of 40N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15cm. How much work is done in stretching the spring from 15cm to 18cm?

$F = kx$

If x is $15-10 = 5$ cm $= 0.05$ m then $F = 40$ N.

$40 = k(0.05)$ $\quad k = \frac{40}{0.05} = 800$

15cm − 10cm = 5 cm = 0.05m

18 cm − 10cm = 8 cm = 0.08m

$$W = \int_{0.05}^{0.08} 800x\,dx = \frac{800x^2}{2}\Big|_{0.05}^{0.08}$$

$= 1.56\ J$

ΔF is constant for each one of the pieces.

Let ΔW_i denote the work required for i^{th} piece (i starting from the top)

$$\text{So Total Work} = \sum_{i=1}^{n} \Delta W_i = \sum_{i=1}^{n} \Delta F_i d_i$$

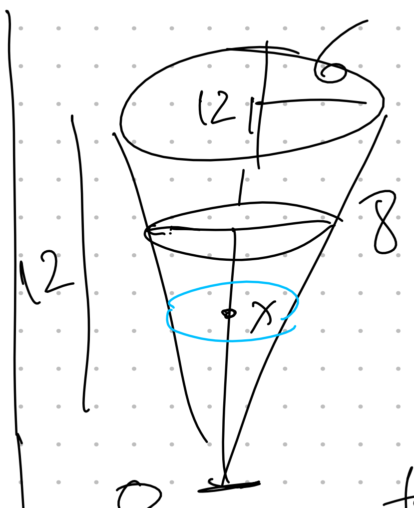
$$= \sum_{i=1}^{n} 2\Delta l\, d_i = \sum_{i=1}^{n} 2\Delta l\, i\Delta l$$

since $d_i = i\Delta l$

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

$$\approx \sum_{i=1}^{n} 2x_i \Delta x \approx \int_0^{100} 2x\,dx$$

$$= x^2 \Big|_0^{100} = 100^2 = 10\,000 \text{ ft-lb}$$



$W = \sum \Delta \text{Work}$

ΔW is the work required to lift the slice at height x to the top of the tank.

Example A tank has the shape of an inverted circular cone with height 12 m and base radius 6m. It is filled with water to a height of 8 meters. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m³.)

$\Delta W = \Delta F\, d \qquad d = 12 - x$

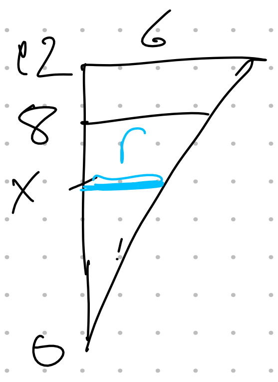
$\Delta F = \Delta \text{Weight} = \Delta \text{Vol} \left(\text{weight density}\right)$

$(g \cdot 1000 \text{ kg}/m^3)$

$\Delta \text{Vol} = \text{Area} \cdot \Delta x$

$= \pi r^2 \Delta x$

what is r in terms of x?



$\dfrac{r}{x} = \dfrac{6}{12} \qquad r = \dfrac{x}{2}$

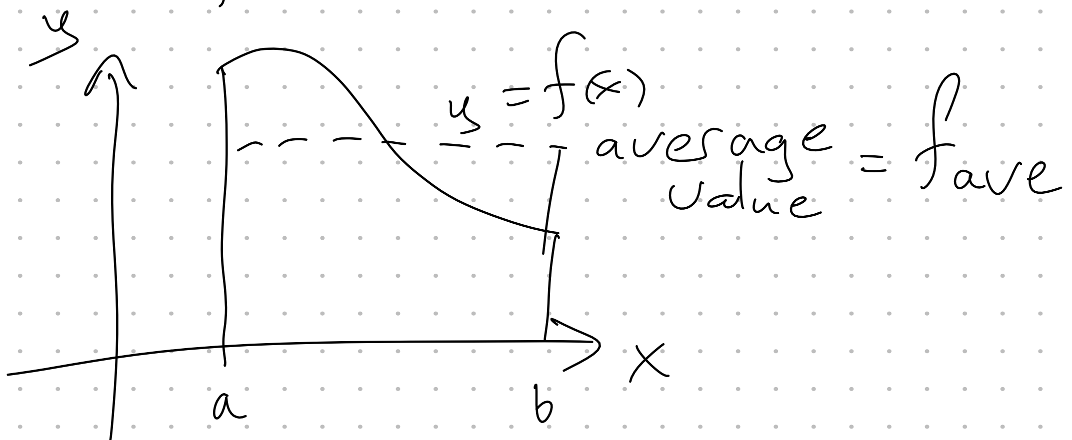
$$\Delta W = \pi r^2 \Delta x \; g \; 1000 (12 - x)$$

$$= \frac{\pi}{4} x^2 g \; 1000 (12 - x) \Delta x$$

$$W = \int_0^8 \frac{\pi}{4} g \; 1000 \; x^2 (12 - x)\,dx = 250\pi g \int_0^8 (12x^2 - x^3)\,dx$$

$$= 250\pi g \left(4x^3 - \frac{x^4}{4}\right) \Big|_0^8 = 250\pi (9.8)\, 2(8^3) \text{ J}$$

6.5 Average Value of a function



We want the area of the rectangle to be the same as the area under the curve.

$$f_{ave}(b - a) = \int_a^b f(x)\,dx$$

$$f_{ave} = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$