Some remarks on the Taylor Series Method: $\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$

- We need to compute several derivatives (possibly by hand) first.
- The function $f(t,x)$ may not have derivatives near $(t_0, x_0)$
- If it is easy to compute several derivatives of $f$ (e.g. $f$ is polynomial of $t$ and $x$), then many terms may be used and high precision can be achieved even with a large value of $h$.

- The Taylor-series method of order $n=1$ is called Euler's method.

$$x(t+h) = x(t) + h x'(t) = x(t) + h f(t,x) \qquad \text{Error } \mathcal{O}(h)$$

## 8.3 Runge-Kutta Methods

Notation:

Say $f = f(x,y)$. Then $f_x = \frac{\partial f}{\partial x}$ $f_y = \frac{\partial f}{\partial y}$

Recall: (Chain Rule in several variables)

Suppose further that $x = x(u)$ and $y = y(u)$. Then

$$f' = \frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} = f_x x' + f_y y'$$

Thus, if $f = f(t,x)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt} = f_t + f_x x'$$

Recall: (Taylor series in two variables)

For $f = f(x,y)$,

$$f(x+h, y+k) = \underbrace{f + h f_x + k f_y + \text{higher order terms}}_{\text{evaluated at } (x,y)}$$

e.g. If $f = f(t,x)$,

$$f(t+\alpha h, x+\beta h f) = \underbrace{f + \alpha h f_t + \beta h f f_x + \text{higher order terms}}_{\text{evaluated at } (t,x)}$$

Example (Second Order)

$$x' = f(t,x)$$
$$x(t_0) = x_0$$

$$x(t+h) = x(t) + h x'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + \cdots$$

$$x' = f(t,x) = f$$
$$x'' = f_t + f_x x'$$
$$= f_t + f_x f$$
$$x''' = f_{tt} + f_{tx}x' + (f_{xt} + f_{xx}x')f + f_x(f_t + f_x x')$$
$$= f_{tt} + f_{tx}f + (f_{xt} + f_{xx}f)f + f_x(f_t + f_x f)$$

Thus,

$$x(t+h) = x + hf + \tfrac{1}{2}h^2(f_t + f_x f) + \mathcal{O}(h^3)$$

$$= x + \tfrac{1}{2}hf + \tfrac{1}{2}hf + \tfrac{1}{2}h^2 f_t + \tfrac{1}{2}h^2 f_x f + \mathcal{O}(h^3)$$

$$(*) \quad = x + \tfrac{1}{2}hf + \tfrac{1}{2}h\left(f + hf_t + hff_x\right) + \mathcal{O}(h^3)$$

Taylor series in 2 variables:

$$f(t+h, x+hf) = f + hf_t + hf f_x + \mathcal{O}(h^2)$$

Thus

$$x(t+h) = x + \tfrac{1}{2}hf + \tfrac{1}{2}hf(t+h, x+hf) + \mathcal{O}(h^3)$$

Thus we can use

$$(**) \quad x(t+h) = x(t) + \tfrac{1}{2}hf(t,x) + \tfrac{1}{2}hf(t+h, x+hf(t,x))$$

for an iteration. More compactly,

$$x(t+h) = x(t) + \tfrac{1}{2}(F_1 + F_2)$$

where
$$F_1 = hf(t,x)$$
$$F_2 = hf(t+h, x+F_1)$$

More generally a second-order Runge-Kutta formula is of the form

$$x(t+h) = x + w_1 hf + w_2 hf(t+\alpha h, x+\beta hf) + \mathcal{O}(h^3)$$

where $w_1, w_2, \alpha, \beta$ are some constant satisfying a system of equation.

Using Taylor series in two variables again we see that

$$x(t+h) = x + w_1 hf + w_2 h\left(f + \alpha h f_t + \beta h f f_x\right) + \mathcal{O}(h^3)$$

Recall $(*)$:

$$x(t+h) = x + \tfrac{1}{2}hf + \tfrac{1}{2}h\left(f + hf_t + hff_x\right) + \mathcal{O}(h^3)$$

$\Rightarrow$ we must have

$w_1 + w_2 = 1 \qquad \alpha w_2 = \frac{1}{2} \qquad \beta w_2 = \frac{1}{2}$

If we choose $w_1 = w_2 = \frac{1}{2}$ $\quad \alpha = \beta = 1$,

we obtain the formula $(**)$

If we choose $w_1 = 0 \quad w_2 = 1 \quad \alpha = \beta = \frac{1}{2}$,

we obtain the modified Euler method:

$x' = f$

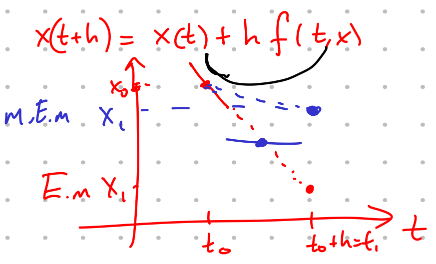
Euler's Method

$x(t+h) = x(t) + h f(t,x)$

$$x(t+h) = x(t) + F_2$$

where $\quad F_1 = h f(t,x)$

$\qquad\qquad F_2 = h f(t + \frac{1}{2}h, x + \frac{1}{2}F_1)$




Example (Fourth-Order Runge-Kutta Method)

$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

where $\quad F_1 = h f(t,x)$

$\qquad\qquad F_2 = h f(t + \frac{1}{2}h, x + \frac{1}{2}F_1)$

$\qquad\qquad F_3 = h f(t + \frac{1}{2}h, x + \frac{1}{2}F_2)$

$\qquad\qquad F_4 = h f(t + h, x + F_3)$

Example Apply the $4^{th}$ order RK method to $x' = \lambda x$.

Note that $f(t,x) = \lambda x$ so

$$F_1 = h f(t,x) = h\lambda x$$

$$F_2 = h f(t + \tfrac{1}{2}h, x + \tfrac{1}{2}F_1) = h\lambda(x + \tfrac{1}{2}h\lambda x) = (h\lambda + \tfrac{1}{2}h^2\lambda^2)x$$

$$F_3 = h\lambda(x + \tfrac{1}{2}F_2) = h\lambda x + \tfrac{1}{2}h\lambda(h\lambda + \tfrac{1}{2}h^2\lambda^2)x = \left(h\lambda + \tfrac{1}{2}h^2\lambda^2 + \tfrac{1}{4}h^3\lambda^3\right)x$$

$$F_4 = h\lambda(x + F_3) = h\lambda x + h\lambda\left(h\lambda + \tfrac{1}{2}h^2\lambda^2 + \tfrac{1}{4}h^3\lambda^3\right)x$$

$$= \left(h\lambda + h^2\lambda^2 + \tfrac{1}{2}h^3\lambda^3 + \tfrac{1}{4}h^4\lambda^4\right)x$$

Thus, $F_1 + 2F_2 + 2F_3 + F_4$

$$= \underline{h\lambda} x + 2(\underline{h\lambda} + \tfrac{1}{2}\underline{h^2\lambda^2})x + 2\left(\underline{h\lambda} + \tfrac{1}{2}\underline{h^2\lambda^2} + \tfrac{1}{4}\underline{h^3\lambda^3}\right)x$$

$$+ \left(\underline{h\lambda} + \underline{h^2\lambda^2} + \tfrac{1}{2}\underline{h^3\lambda^3} + \tfrac{1}{4}h^4\lambda^4\right)x$$

$$= \left[6h\lambda + 3h^2\lambda^2 + h^3\lambda^3 + \tfrac{1}{4}h^4\lambda^4\right]x$$

Thus, $x(t+h) = x + \frac{1}{6}(F_1 + 2F_2 + F_3 + F_4)$

$$x(t+h) = \left[1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right]x(t)$$

Note that analytic solutions of $x' = \lambda x$ are given by $x(t) = Ce^{\lambda t}$. Thus,

$$x(t+h) = Ce^{\lambda(t+h)} = Ce^{\lambda t}.e^{\lambda h} = x(t)\,e^{\lambda h}$$

$$= \left(1 + \lambda h + \frac{1}{2!}\lambda^2h^2 + \frac{1}{3!}\lambda^3h^3 + \frac{1}{4!}\lambda^4h^4 + \cdots\right)x(t)$$

Common Mistakes from HW4 and HW5

1) $A, B$ $n\times n$ matrices. $\qquad \|A\|\|B\| \neq \|AB\|$

$\|AB\| \leq \|A\|\|B\|$ is typically not an equality!

2) $\|I - A\| < 1 \Rightarrow A$ is invertible.

However, the converse is not true:

$A$ is invertible $\not\Rightarrow \|I - A\| < 1$

e.g. $A = 101I$ then $A$ is invertible $\left(A^{-1} = \frac{1}{101}I\right)$

but $\quad \|I - A\| = \|-100I\| = 100\|I\| = 100 > 1$.

3) When we want to find a SVD for a matrix $A_{m\times n}$

$(A = PDQ) \qquad A_{m\times n} = P_{m\times m} D_{m\times n} Q_{n\times n} \qquad$ P and Q are square

First, we find $n$-eigenvectors of $(A^*A)_{n\times n}$ that are orthonormal

In particular, we choose all of them as unit vectors. Call them $u_1, u_2, \dots, u_n$. (They completely define $Q$.)

Then we set $v_i = \sigma_i^{-1} A u_i \in \mathbb{C}^m$ for $i \leq r$

($\sigma_i \neq 0$ and $\sigma_i = 0$ for $i > r$.) $\qquad r = \text{rank}(A) \leq \min(m,n)$

$P$ needs to be an $m\times m$ matrix so $v_1, v_2 \cdots v_r$ is not enough to completely determine $P$ if $r < m$.

In that case, we have to extend $v_1, v_2, \cdots v_r$ to an orthonormal basis for $\mathbb{C}^m$. (Find $(m-r)$ new unit vectors and every pair of vectors $v_1, \cdots v_r, \cdots v_m$ is orthogonal.)