Chp 8 Numerical Solution of O.D.E.s

8.1 The Existence and Uniqueness of Solutions

An eq. of the form $x' = f(t,x)$ is called an ordinary differential equation. (o.d.e.) Here x represents an unknown function of t. Thus, a solution $x(t)$ satisfies:

$$\frac{dx(t)}{dt} = f(t, x(t))$$

An Initial-Value Problem (IVP) is an o.d.e. with "initial value" that is

$$x' = f(t,x)$$
$$x(t_0) = x_0$$

Example

$$x' = x\tan(t+3)$$
$$x(-3) = 1$$

Analytic solution can be found using "seperation of variables":

$$\frac{1}{x}\frac{dx}{dt} = \tan(t+3) \Rightarrow \int \frac{1}{x}\frac{dx}{dt}\,dt = \int \tan(t+3)\,dt$$

$$\Rightarrow \int \frac{1}{x}\,dx = \int \frac{\sin(t+3)}{\cos(t+3)}\,dt \quad \left(\begin{array}{l}\text{using } u = x(t) \text{ subs. for the LHS}\\ \text{and } u = \cos(t+3) \text{ subs. for the RHS.}\end{array}\right.$$

$$\Rightarrow \ln|x| = \ln|\sec(t+3)| + C$$

$e^C = K$

$$e^{\ln|x|} = |x| = e^C|\sec(t+3)| = K|\sec(t+3)|$$

Since $t_0 = -3$, $x_0 = 1$, we can assure $x > 0$ and

$x = K\sec(t+3)$ gives us

$$1 = K\sec(-3+3) = K \Rightarrow x(t) = \sec(t+3)$$

Very often we cannot find the analytic solution and we have to use numerical methods. Before that we need to ask

Q1) Do IVPs always have solutions?

Q2) If there is a solution, is it unique?

A1) Not in general. Even if a solution exists, the solution may exists only in a neighborhood of the initial value (t_0, x_0).

e.g.

$$x' = 1 + x^2 \qquad x(0) = 0$$

$$\Rightarrow \int \frac{1}{1+x^2}\,dx = \int dt \Rightarrow \tan^{-1}(x) = t + C$$

$\Rightarrow x = \tan(t+C)$. Since $x(0) = 0$, $C = 0$ and $x = \tan(t)$.

Thus, x is not defined at $t = \pm\frac{\pi}{2}$. Domain of x is $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Thm (Existence Theorem 1)

If f is cont. in a rectangle R centered at (t_0, x_0)

$$R = \{(t,x) \mid |t - t_0| \le \alpha,\ |x - x_0| \le \beta\}$$

then the IVP $\quad x' = f(t,x)$, $x(t_0) = x_0$

has a solution $x(t)$ for $|t - t_0| \le \min(\alpha, \beta/M)$

where $M = \max\limits_{(t,x)\in R} |f(t,x)|$

e.g. Prove that the IVP $\quad x' = (t + \sin x)^2 \qquad f(t,x) = (t+\sin x)^2$

$x(0) = 3 \qquad t_0 = 0 \quad x_0 = 3$

has a solution on $-1 \le t \le 1$.

Consider $R = \{(t,x) \mid |t| \le \alpha,\ |x - 3| \le \beta\}$

$$|f| = (t+\sin x)^2 \le (\alpha+1)^2 \qquad \text{Thus, } M = \max_R |f| \le \underbrace{(\alpha+1)^2}_{}\overset{\beta}{}$$

Choosing $\alpha = 1$ and $\beta = (\alpha+1)^2$, we get $\beta/M \ge 1$ and a solution $x(t)$ defined for $|t| \le \min(\alpha, \beta/M) = \min(1, \beta/M) = 1$.

A2) Not in general but we can impose some conditions on f to get uniqueness.

$f(t,x) = 3x^{2/3}$ is cont in both t and x

e.g. $\qquad x' = 3x^{2/3} \qquad x(0) = 0. \qquad (t_0, x_0) = (0,0)$

Clearly, $x(t) = 0$ and $x(t) = t^3$ are two distinct solutions to the IVP.

$$x' = 3t^2 = 3(t^3)^{2/3} = 3x^{2/3}$$

f is cont but $f_x = \frac{\partial f}{\partial x} = 2x^{-1/3}$ is not defined at $(t_0, x_0) = (0,0)$

Thm (Existence and Uniqueness)

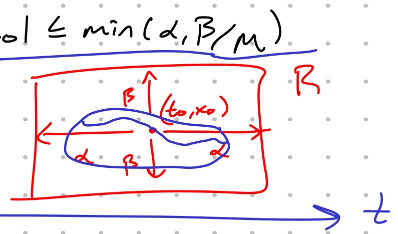
If f and $f_x = \frac{\partial f}{\partial x}$ are cont. on

$$R = \{(t,x) \mid \boxed{|t - t_0| \le \alpha},\ |x - x_0| \le \beta\}$$

then the IVP $\quad x' = f(t,x)$, $x(t_0) = x_0 \quad$ has

a unique solution $x(t)$ for $|t - t_0| \le \min(\alpha, \beta/M)$

where $M = \max\limits_{(t,x)\in R} |f(t,x)|$



Def^n f is called Lipschitz continuous on I if $\exists L$ s.t. $|f(x) - f(y)| \le L|x-y| \quad \forall\, x,y \in I$.

Exercise: Prove that if f is Lipschitz cont. on I then it is cont. on I, that is L. cont. fns $\subseteq$ cont. fns.

Thm (Existence and Uniqueness 2)

If $f(t,x)$ continuous on $[a,b]\times\mathbb{R}$ and Lipschitz cont. in x in the sense that:

$$|f(t,x) - f(t,y)| \le L|x-y| \qquad \forall\, t\in[a,b],\ x,y\in\mathbb{R}$$

then the IVP $\begin{array}{l} x' = f(t,x) \\ x(t_0) = x_0 \end{array}$ has a unique solution $x(t)$ for $t \in [a,b]$.

Note that if f' exists and $|f'(x)| \le L \quad \forall x \in I$ then for $x_1, x_2 \in I \quad \exists\, x_1 < y < x_2$ s.t.

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(y) \quad \text{by the MVT.}$$

Thus, $|f(x_1) - f(x_2)| = |f'(y)|\,|x_1 - x_2| \le L|x_1 - x_2| \quad \forall x_1, x_2 \in I$.

That is f is Lipschitz cont. Thus,

fns with bounded derivative $\subset$ L. cont. fns. $\subset$ cont. fns.

8.2 Taylor Series Method

In numerical solutions of $\begin{array}{l} x' = f(t,x) \\ x(t_0) = x_0 \end{array}$, we typically do not expect formulas directly. Instead we first construct a table of values

Given | — computed

t_0	t_1	t_2	$\cdots$	t_n
x_0	x_1	x_2	$\cdots$	x_n

and then use approximating functions (e.g. spline functions) to get approximations in between.

We will denote the exact solution at t_i by $x(t_i)$ and x_i denotes an approximation to $x(t_i)$.

Example (Taylor-Series Method)

Given $\begin{cases} x' = \cos t - \sin x + t^2 \\ x(-1) = 3 \end{cases}$, we compute

$$x'' = -\sin t - x'\cos(x) + 2t$$

$$x''' = -\cos t - x''\cos(x) + (x')^2 \sin x + 2$$

$$x^{(4)} = \sin t - x'''\cos x + 3x'x''\sin x + (x')^3 \cos x$$

When taking derivative of the RHS, we use chain rule e.g.

$$\frac{d}{dt}(-\sin x) = \frac{d}{dt}(-\sin(x(t))) = -\cos(x(t))\cdot\frac{dx}{dt}(t)$$

$$x(t+h) = x(t) + h\,x'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + \frac{h^4}{4!}x^{(4)}(t) + \underbrace{O(h^5)}_{\text{truncation error}}$$

Setting $h = 0.01$, the above formulas help us compute $x(-1+h) = x(-0.99)$, $\quad t_0 = -1 \quad x_0 = x(t_0) = 3$

$$x(-1+2h) = x(-0.99+h) = x(-0.98)$$

$$x(-1+3h) = x(-0.98+h) = x(-0.97)$$

and so on.

An algorithm that accomplishes this is:

```
input M ← 200, h ← 0.01, t ← -1.0, x ← 3.0
output 0, t, x
for k = 1 to M do
    x'  ← cos t - sin x + t^2
    x'' ← -sin t - x' cos(x) + 2t
    x''' ← -cos t - x'' cos(x) + (x')^2 sin x + 2
    x^(4) ← sin t - x''' cos x + 3x'x'' sin x + (x')^3 cos x
    x ← x + h(x' + h/2 (x'' + h/3 (x''' + h/4 x^(4))))
    t ← t + h
    output k, t, x
end do
```

This gives us

k	t	x
0	−1.00	3.000
1	−0.99	3.014
2	−0.98	3.028
3	−0.97	3.042
⋮	⋮	⋮
198	0.98	6.394
199	0.99	6.408
200	1.00	6.422

Q) Is the approximation $x(1.00) \approx x_{200} = 6.422$ reliable?

A) In fact, if we ran the algorithm starting at $(1, 6.422)$ with $h = -0.01$, we get $x(-1.00) \approx x_{200} = 3.00$. Thus, in this case, the approximations appear to be good.

What can we say about the error, in general?

There are 4 types of error involved in this calculation:

1) Local truncation error
2) Local round off error
3) Global truncation error
4) Global round off error

Local refers to 1 iteration of the for loop.

e.g. Local truncation error is the error in approximating $x(t+h)$ by $x(t) + h\,x'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + \frac{h^4}{4!}x^{(4)}(t)$ even if everything else was computed exactly.

Local round off error is the error due to the roundoff in computing $x(t) + h\,x'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + \frac{h^4}{4!}x^{(4)}(t)$

Global X error is the error accumulated due to local X errors in all the iterations of the for loop.

In our example, since the order of the Taylor series method is 4, the local truncation error is $O(h^5)$

Since the number of iterations of the loop is proportional to $\frac{1}{h}$, the global truncation error is $O(h^5 \cdot \frac{1}{h}) = O(h^4)$

because the last term we use involves $x^{(4)}$