Last time:

Set $A_i = \int_a^b \ell_i(x)\,dx$. Then $\int_a^b f(x)\,dx \overset{\otimes}{\approx} \sum_{i=0}^{n} A_i f(x_i)$

If $x_0, x_1, \dots, x_n$ are equally spaced, $\otimes$ is called a Newton-Cotes formula.

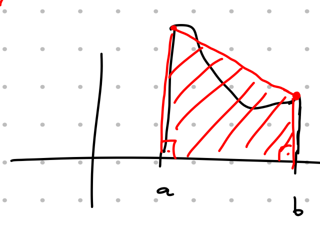
Trapezoid Rule

If $n=1$, $x_0 = a$, $x_1 = b$, then $A_0 = A_1 = \frac{1}{2}(b-a)$

Thus the quadrature formula is

a formula that gives the area

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right]$$

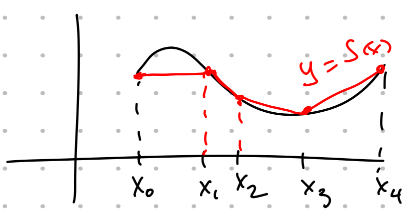


$$\left(\text{Error} = -\frac{1}{12}(b-a)^3 f''(y) \qquad y \in (a,b)\right)$$

If $a = x_0 < x_1 < \dots < x_n = b$, we can apply the trapezoid rule to each one of the subintervals $[x_i, x_{i+1}]$ to obtain the composite trapezoid rule:

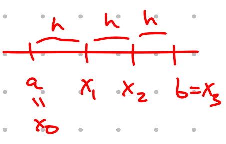
$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{1}{2}\sum_{i=0}^{n}(x_{i+1}-x_i)\left(f(x_i)+f(x_{i+1})\right)$$

We also obtain the composite trapezoid rule if we replace f with a spline function of degree $= 1$.



$$\int_a^b f(x)\,dx \approx \int_a^b S(x)\,dx$$

With uniform spacing: $h = \frac{b-a}{n}$ $\quad x_i = a + ih$,



$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right)$$

$$\left(\text{Error} = -\frac{1}{12}(b-a)h^2 f''(y)\right)$$

Exercise: Show that the Newton-Cotes formula with $n=2$ on $[0,1]$ is

$$\int_0^1 f(x)\,dx \approx \frac{1}{6}f(0) + \frac{2}{3}f\left(\tfrac{1}{2}\right) + \frac{1}{6}f(1)$$

find x_0, x_1, x_2 ($= 0, \frac{1}{2}, 1$)

find ℓ_0, ℓ_1, ℓ_2

find A_0, A_1, A_2. $A_i = \int_0^1 \ell_i\,dx$

Method of Undetermined Coefficients

Note that $\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$ is exact if $f(x)$ is a polynomial of degree $\le n$.

This allows us to determine the coefficients A_i without evaluating $A_i = \int_a^b \ell_i(x)\,dx$.

Example:

Find A_0, A_1, A_2 in Newton-Cotes formula with $n=2$ on $[0,1]$.

Since the formula $\int_0^1 f(x)\,dx \overset{*}{\approx} \sum_{i=0}^{2} A_i f(x_i)$ (where $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$) is exact for poly. of deg. ≤ 2, we have

for $f(x) = 1$: $\quad 1 = \int_0^1 1\,dx = A_0 + A_1 + A_2$

for $f(x) = x$: $\quad \frac{1}{2} = \int_0^1 x\,dx = \frac{1}{2}A_1 + A_2$

for $f(x) = x^2$: $\quad \frac{1}{3} = \int_0^1 x^2\,dx = \frac{1}{4}A_1 + A_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{4} & 1 & \frac{1}{3} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{12} \end{array}\right) \Rightarrow \frac{1}{2}A_2 = \frac{1}{12} \qquad A_2 = \frac{1}{6}$$

$$\Rightarrow A_1 + \frac{1}{3} = 1 \qquad A_1 = \frac{2}{3} \qquad \Rightarrow A_0 + \frac{2}{3} + \frac{1}{6} = 1 \qquad A_0 = \frac{1}{6}$$

Thus, $\int_0^1 f(x)\,dx \approx \frac{1}{6}f(0) + \frac{2}{3}f\left(\frac{1}{2}\right) + \frac{1}{6}f(1)$

$\cos(0)$, $\cos(\frac{\pi}{4})$, $\cos(\frac{\pi}{2}) = 0$

$$\int_0^1 \cos\left(\frac{\pi x}{2}\right)dx \approx \frac{1}{6} + \frac{2}{3}\cdot\frac{\sqrt{2}}{2} = \frac{2\sqrt{2}+1}{6} \approx 0.638071$$

$$\int_0^1 \cos\left(\frac{\pi x}{2}\right)dx = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\Big|_0^1 = \frac{2}{\pi} \approx 0.636620$$

Thus $|\text{error}| \approx 0.001451$ $\qquad$ $|\text{rel. error}| \approx 0.227988\ \%$

General Integration Formulas

In general we may consider integration formulas of type

$$\int_a^b f(x)\,w(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

where w is a fixed weight function.

Here $\quad A_i = \int_a^b \ell_i(x)\,w(x)\,dx$

Example Find a formula

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx \approx A_0 f\left(-\frac{3\pi}{4}\right) + A_1 f\left(\frac{-\pi}{4}\right) + A_2 f\left(\frac{\pi}{4}\right) + A_3 f\left(\frac{3\pi}{4}\right)$$

that is exact when f is a poly. of deg. ≤ 3.

basis for P_3
$\{1, x, x^2, x^3\}$
$\{\ell_0, \ell_1, \ell_2, \ell_3\}$

Note that for $f(x) = x$ we have

$$0 = \int_{-\pi}^{\pi} \underbrace{x\cos(x)}_{\text{odd func.}}\,dx = \underbrace{(A_3 - A_0)}_{x}\frac{3\pi}{4} + \underbrace{(A_2 - A_1)}_{y}\frac{\pi}{4} = 0$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and for $f(x) = x^3$ we have

$$0 = \int_{-\pi}^{\pi} \underbrace{x^3\cos(x)}_{\text{odd func.}}\,dx = \underbrace{(A_3 - A_0)}_{x}\left(\frac{3\pi}{4}\right)^3 + \underbrace{(A_2 - A_1)}_{y}\left(\frac{\pi}{4}\right)^3 = 0$$

These two equations $\Rightarrow$ $A_3 = A_0$ $\quad A_2 = A_1$.

for $f(x) = 1$,

$$0 = \int_{-\pi}^{\pi} \cos(x)\,dx = 2A_0 + 2A_1 \quad \Rightarrow \quad A_1 = -A_0.$$

Finally, $f(x) = x^2$ gives us

integration by parts (twice)

$$-4\pi = \int_{-\pi}^{\pi} x^2 \cos x\,dx = 2A_0\left(\frac{3}{4}\pi\right)^2 - 2A_0\left(\frac{\pi}{4}\right)^2$$

$$\Rightarrow A_0 = \frac{-4\pi}{\frac{9}{8}\pi^2 - \frac{\pi^2}{8}} = \frac{-4}{\pi} = A_3 = -A_1 = -A_2$$

Thus,

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx \approx -\frac{4}{\pi}\left(f\left(-\frac{3\pi}{4}\right) - f\left(\frac{-\pi}{4}\right) - f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{4}\right)\right)$$

7.3 Gaussian Quadrature

We are again interested in formulas of type

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad \text{which are exact for } f \in P_n$$

(P_n: the space of poly.s of deg. $\leq n$ the textbook uses Π_n for P_n)

Q) Are there "good choices" of x_i s?

For example, we might want to look for $x_0, x_1, \dots, x_n$ s.t.

$$A_0 = A_1 = \dots = A_n = c \quad \text{leading to}$$

$$\int_a^b f(x)\,dx \approx c\sum_{i=0}^{n} f(x_i)$$

e.g. Say we are looking for $\alpha, \beta \in [-1, 1]$ s.t.

$$\int_{-1}^{1} f(x)\,dx \approx c(f(\alpha) + f(\beta))$$

Then, $f(x) = 1$ gives us $\quad 2 = c(1+1) \quad \Rightarrow c = 1$

$f(x) = x$ gives us $\displaystyle\int_{-1}^{1} x\,dx = 0 = f(\alpha) + f(\beta) = \alpha + \beta \Rightarrow \alpha = -\beta$

$f(x) = x^2$ gives us $\displaystyle\int_{-1}^{1} x^2\,dx = \frac{2}{3} = \alpha^2 + \beta^2 = 2\alpha^2 \Rightarrow \alpha = \pm\frac{1}{\sqrt{3}}$

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

e.g. $(n = 4)$

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{5}\left(f(-\alpha) + f(-\beta) + f(0) + f(\beta) + f(\alpha)\right)$$

where $\alpha = \sqrt{\frac{5+\sqrt{11}}{12}} \quad \beta = \sqrt{\frac{5-\sqrt{11}}{12}}$

Gaussian Quadrature

We know we can choose A_i so that

$$\int_a^b f(x)w(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad \text{is exact for } f \in P_n$$

($w(x)$ is a positive weight function.)

By changing the nodes x_i we use, we can get exact results for $f \in P_{2n+1}$!

Thm (Gaussian Quadrature)

let w be a positive weight function and let $q \in P_{n+1}$ be non-zero that is w-orthogonal to P_n in the sense that

$$\int_a^b q(x)p(x)w(x)\,dx = 0 \qquad \forall p \in P_n.$$

If $x_0, x_1, \dots, x_n$ are the zeros of q, then

$$(*) \quad \int_a^b f(x)w(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad \text{is exact for } f \in P_{2n+1}$$

where $\displaystyle A_i = \int_a^b w(x)\,\ell_i(x)\,dx.$

Pf let $f \in P_{2n+1}$. Then $\exists!\, p, r \in P_n$ s.t. $\overset{\deg 2n+1}{f} = \overset{n}{p}\,\overset{n+1}{q} + \overset{n}{r}$

Thus, $f(x_i) = p(x_i)\underbrace{q(x_i)}_{=0} + r(x_i) = r(x_i)$. Since $(*)$ is exact for $r \in P_n$,

$$\int_a^b fw\,dx = \int_a^b (pqw + rw)\,dx = \underbrace{\int_a^b pqw\,dx}_{=0} + \int_a^b rw\,dx = \sum_{i=0}^{n} A_i r(x_i) = \sum_{i=0}^{n} A_i f(x_i) \quad \square$$

Thm (on Number of Sign Changes)

Let w be a positive weight function in $C[a,b]$. Let $0 \neq q \in C[a,b]$ that is w-orthogonal to P_n. Then q changes sign at least $n+1$ times on (a,b). $\boxed{\int_a^b pqw\,dx = 0 \ \ \forall p \in P_n}$

Pf Since $1 \in P_n$ and q is w-orthogonal to P_n,

$$\int_a^b 1 \, q(x) w(x) \, dx = 0 \Rightarrow q(x) \text{ changes sign at least once.}$$

Suppose q change sign only $r \le n$ times. Choose t_i s.t.

$$a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b \quad \text{and}$$

q does not change sign on (t_i, t_{i+1}) $0 \le i \le r$

Set $p(x) = (x-t_1)(x-t_2)\cdots(x-t_r)$

Then (1) $p(x) \in P_n$

(2) $p(x)q(x)$ is either ≥ 0 $\forall x \in [a,b]$
or ≤ 0 $\forall x \in [a,b]$

Since $w(x) > 0$ $\forall x$,

$$\int_a^b p(x)q(x)w(x)\,dx \neq 0 \quad \text{which is a contradiction.} \quad \square$$

The properties:

1) q_{n+1} is a monic polynomial

2) q_{n+1} is w-orthogonal to P_n

uniquely determines $q_{n+1} \in P_{n+1}$.

For efficiency we use the following thm from section 6.8.

Thm 5 (Section 6.8)

The sequence of polynomials defined by

$$p_n(x) = (x - a_n)\, p_{n-1}(x) - b_n\, p_{n-2}(x) \qquad (n \ge 2)$$

with $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$a_n = \frac{\langle x p_{n-1}, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} \qquad \text{where } \langle f, g \rangle = \int_a^b f g w \, dx$$

$$b_n = \frac{\langle x p_{n-1}, p_{n-2} \rangle}{\langle p_{n-2}, p_{n-2} \rangle}$$

is orthogonal. $\langle p_n, p_m \rangle = 0$ if $n \neq m$.

(let $w(x) = 1$, $a = -1$, $b = 1$)

Thus, $q_0(x) = 1$ $\quad q_1(x) = x - a_1$

$$a_1 = \frac{\langle x q_0, q_0 \rangle}{\langle q_0, q_0 \rangle} = \int_{-1}^1 x\,dx \Big/ \int_{-1}^1 dx = 0$$

so $q_1(x) = x$

$$q_2(x) = (x - a_2)\, q_1(x) - b_2\, q_0(x) = x^2 - a_2 x - b_2$$

$$a_2 = \frac{\langle x q_1, q_1 \rangle}{\langle q_1, q_1 \rangle} = \int_{-1}^1 x^3 dx \Big/ \langle q_1, q_1 \rangle = 0$$

$$b_2 = \frac{\langle x q_1, q_0 \rangle}{\langle q_0, q_0 \rangle} = \int_{-1}^1 x^2 dx \Big/ \int_{-1}^1 dx = \frac{2/3}{2} = \frac{1}{3}$$

so $q_2(x) = x^2 - \frac{1}{3}$ and so on.

Note that for $n = 1$, corrections $q_{n+1}(x) = x^2 - \frac{1}{3}$ so its roots are $\pm \frac{1}{\sqrt{3}}$

and we get the formula $\int_{-1}^1 f(x)\,dx \approx A_0 f\left(-\frac{1}{\sqrt{3}}\right) + A_1 f\left(\frac{1}{\sqrt{3}}\right)$

since it has to be exact for $f(x) \in P_1$, correction P_3, this reduces

to the earlier example $\int_{-1}^1 f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ (We see that this is exact for $f \in P_3$ correction)

Exercise: Compute $q_3(x)$. $\left(q_3(x) = x^3 - \frac{3}{5}x\right)$

Its roots are $0, \pm\sqrt{\frac{3}{5}}$. Thus,

$$\int_{-1}^1 f(x)\,dx \approx A_0 f\left(-\sqrt{\tfrac{3}{5}}\right) + A_1 f(0) + A_2 f\left(\sqrt{\tfrac{3}{5}}\right).$$

Determine A_0, A_1, A_2 using the method of undetermined coefficients.