6.2 Divided Differences

Let $x_0, x_1, \dots, x_n$ be a set of nodes

and $f(x_0), f(x_1), \dots, f(x_n)$ be the values of a function at those nodes.

Set $q_0(x) = 1$

$q_1(x) = (x - x_0)$

$q_2(x) = (x - x_0)(x - x_1)$

$\vdots$

$q_n(x) = (x - x_0)(x - x_1)\cdots(x - x_{n-1})$

Then the Newton form of an interpolation polynomial is

$$p(x) = \sum_{j=0}^{n} c_j q_j(x) \qquad n+1 \text{ variables}$$

$$p(x_i) = f(x_i) \qquad 0 \le i \le n \qquad n+1 \text{ equations}$$

$$\Rightarrow \sum_{j=0}^{n} c_j q_j(x_i) = f(x_i)$$

$$\begin{pmatrix} q_0(x_0) & q_1(x_0) & \cdots & q_n(x_0) \\ q_0(x_1) & q_1(x_1) & \cdots & q_n(x_1) \\ \vdots & \vdots & & \\ q_0(x_n) & q_1(x_n) & \cdots & q_n(x_n) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ \vdots \\ \vdots \\ f(x_n) \end{pmatrix}$$

Note that $q_j(x_i) = (x_i - x_0)(x_i - x_1)\cdots(x_i - x_{j-1})$

so if $i < j$, $q_j(x_i) = 0$

Thus, (lower triangular)

$$\begin{pmatrix} q_0(x_0) & 0 & \cdots & 0 \\ q_0(x_1) & q_1(x_1) & 0 & \cdots 0 \\ \vdots & \vdots & & \vdots \\ & & & 0 \\ q_0(x_n) & q_1(x_n) & \cdots & q_n(x_n) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ \vdots \\ \vdots \\ f(x_n) \end{pmatrix}$$

eg. $(n=2)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & x_1 - x_0 & 0 \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}$$

can be solved using forward subs.

"Divided differences"

$\Rightarrow c_0 = f(x_0)$

$$f(x_0) + (x_1 - x_0)c_1 = f(x_1) \quad \Rightarrow \quad c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_0) + (x_2 - x_0)\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right) + (x_2 - x_0)(x_2 - x_1)c_2 = f(x_2)$$

before we solve the last equation, new notation:

$$c_0 = f[x_0] = f(x_0) \qquad c_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

back to solving for c_2:

$$\Rightarrow (x_2 - x_0)(x_2 - x_1)c_2 = f(x_2) - f(x_0) - (x_2 - x_0)\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)$$

$$\Rightarrow c_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_2 - x_1)(x_1 - x_0)} = \frac{f[x_0, x_2] - f[x_0, x_1]}{x_2 - x_1}$$

$$c_2 = \frac{(f(x_2) - f(x_0))(x_1 - x_0) - (f(x_1) - f(x_0))(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{f(x_2)x_1 - f(x_2)x_0 - f(x_0)x_1 + f(x_0)x_0 - f(x_1)x_2 + f(x_1)x_0 + f(x_0)x_2 - f(x_0)x_0}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{f(x_2)x_1 - f(x_2)x_0 - f(x_0)x_1 + f(x_1)x_1 - f(x_1)x_2 + f(x_1)x_0 + f(x_0)x_2 - f(x_1)x_1}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{f(x_2)(x_1 - x_0) - f(x_1)(x_1 - x_0) - \left(f(x_1)(x_2 - x_1) - f(x_0)(x_2 - x_1)\right)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{(f(x_2) - f(x_1))(x_1 - x_0) - (f(x_1) - f(x_0))(x_2 - x_1)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = c_2$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

Set $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

In particular, we have seen that $f[x_1, x_0, x_2] = f[x_0, x_1, x_2]$.

We've seen that $c_0 = f[x_0]$, $c_1 = f[x_0, x_1]$, $c_2 = f[x_0, x_1, x_2]$.

In general we set $c_n = f[x_0, x_1, \dots, x_n]$ in

$$p(x) = p_n(x) = \sum_{j=0}^{n} c_j q_j(x)$$

Note that $q_n(x)$ is the only polynomial (among $q_0, q_1, \cdots, q_n$) of degree n.

Thus, $c_n = f[x_0, \cdots, x_n]$ is also the coefficient of x^n in $p_n(x)$.

$f[x_0, \cdots, x_n]$ are called divided differences.

Note $p(x) = \sum_{k=0}^{n} f[x_0, \cdots, x_k] \prod_{j=0}^{k-1} (x - x_j)$

Thm 1

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$$

(We have already seen this for $f[x_0, x_1, x_2]$ with direct calculation).

Pf p_k : the polynomial that interpolates f at $x_0, x_1, \cdots, x_k$

q : // // $x_1, \cdots, x_n$

$\deg p_k \le k$ and $\deg q \le n-1$

Consider the polynomial

$$r(x) = q(x) + \frac{x - x_n}{x_n - x_0} \left(q(x) - p_{n-1}(x) \right)$$

p_{n-1}, q interpolate f at n points ($x_0\ x_1\ x_2\ \cdots\ x_i\ \cdots\ x_{n-1}\ x_n$)

$$r(x_0) = q(x_0) + \frac{x_0 - x_n}{x_n - x_0} \left(q(x_0) - p_{n-1}(x_0) \right) = q(x_0) - \left(q(x_0) - p_{n-1}(x_0) \right)$$

$$= p_{n-1}(x_0) = f(x_0)$$

for $1 \le i \le n-1$, $r(x_i) = q(x_i) + \frac{x_i - x_n}{x_n - x_0} \left(q(x_i) - p_{n-1}(x_i) \right)$

$$= f(x_i) + \frac{x_i - x_n}{x_n - x_0} \left(f(x_i) - f(x_i) \right) = f(x_i)$$

$$r(x_n) = q(x_n) + \frac{x_n - x_n}{x_n - x_0} \left(q(x_n) - p_{n-1}(x_n) \right) = f(x_n)$$

That is, r is a poly. (of degree $\le n$) that interpolates f at $x_0, x_1, \cdots, x_n$

$\Rightarrow r = p_n$ by Thm 1 of Section 6.1.

In p_n, the coeff. of x^n is $f[x_0, x_1, \cdots, x_n]$.

In $r(x) = q(x) + \frac{x - x_n}{x_n - x_0} \left(q(x) - p_{n-1}(x) \right)$

$q(x)$: degree $\le n-1$; $q(x)$: coeff of x^{n-1} is $f[x_1, \cdots, x_n]$; $p_{n-1}(x)$: coeff of x^{n-1} is $f[x_0, \cdots, x_{n-1}]$

Thus, the coeff of x^n in r is $\dfrac{f[x_1, \cdots x_n] - f[x_0, \cdots x_{n-1}]}{x_n - x_0}$.

Since $r = p_n$, we are done.

In general,

$$f[x_i, x_{i+1}, \cdots, x_{i+j}] = \frac{f[x_{i+1}, \cdots, x_{i+j}] - f[x_i, \cdots, x_{j-1}]}{x_{i+j} - x_i}$$

Thus, we can compute coefficients as follows:

given | to be computed

	c_0	c_1	c_2	c_3
x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	$f[x_1]$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
x_2	$f[x_2]$	$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example

x	2	3	5
y	2	8	0

Given

$x_0 = 2$: 2, $\frac{8-2}{3-2} = 6$, $\frac{-4-6}{5-2} = -\frac{10}{3}$

$x_1 = 3$: 8, $\frac{0-8}{5-3} = -4$

$x_2 = 5$: 0

$$p(x) = 2 + 6(x-2) - \frac{10}{3}(x-2)(x-3)$$

$p(2) = 2$

$p(3) = 2 + 6 = 8$

$p(5) = 2 + 6(3) - \frac{10}{3}(3 \cdot 2)$

$= 0$

Algorithm:

$(x_0, \cdots x_n, f(x_0), \cdots, f(x_n))$ are input

```
for i=0 to n do
    d_i <- f(x_i)
end do
for j=1 to n do
    for i=n to j step -1 do
        d_i <- (d_i - d_{i-1})/(x_i - x_{i-j})
    end do
end do
```

e.g. given: 2 2 / 3 8 / 5 0

initially $d \to d \to d$ | final state

2 → 2 → 2 | 2

8 → 6 → 6 | 6

0 → −4 → $-\frac{10}{3}$ | $-\frac{10}{3}$

starting from bottom

Thm 2 If $(z_0, z_1, \cdots, z_n)$ is a permutation of $(x_0, \cdots x_n)$

Then $f[z_0, \cdots z_n] = f[x_0, \cdots x_n]$ $\left(f[x_0,x_1,x_2] = f[x_1,x_0,x_2] \right)$

Pf Let p be the poly. (of lowest degree) that interpolates f at $z_0, \cdots z_n$

q // // at $x_0, \cdots x_n$

$\Rightarrow p = q$. Comparing coeff.s of x^n gives us the result.

6.4 Spline Interpolation

$x_0, x_1, \cdots x_n$ (nodes)

Given $t_0 < t_1 < \cdots < t_n$ called "knots",

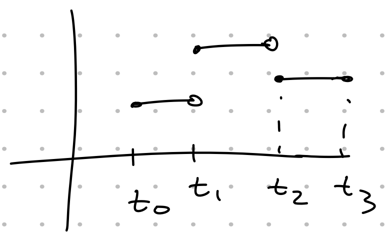
a spline function of degree k is a function S s.t.

1) $S \in C^{k-1}[t_0, t_n]$.

2) On $[t_i, t_{i+1})$, S is poly. of degree $\le k$

e.g. $k = 0$ $(k-1 = -1)$

$$S(x) = \begin{cases} c_0 & \text{if } t_0 \le x < t_1 \\ c_1 & \text{if } t_1 \le x < t_2 \\ \vdots & \vdots \\ c_{n-1} & \text{if } t_{n-1} \le x \le t_n \end{cases}$$



e.g. $k = 1$ $(k-1 = 0 \Rightarrow S$ is cont$)$

$$S(x) = \begin{cases} a_0x + b_0 & \text{if } t_0 \le x < t_1 \\ a_1x + b_1 & \text{if } t_1 \le x < t_2 \\ \vdots & \vdots \\ a_{n-1}x + b_{n-1} & \text{if } t_{n-1} \le x \le t_n \end{cases}$$