Recall Newton form of the interpolation polynomial

$$p_n(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \cdots + c_n(x-x_0)\cdots(x-x_{n-1})$$

Theorem 1

Given distinct $x_0, x_1, \dots, x_n \in \mathbb{R}$ and arbitrary $y_0, y_1, \dots, y_n \in \mathbb{R}$

$\exists!$ polynomial p_n of degree $\leq n$ s.t. $p(x_i) = y_i$

Lagrange Form of the Interpolation Polynomial

$$p_n(x) = y_0\ell_0(x) + y_1\ell_1(x) + \cdots + y_n\ell_n(x) = \sum_{k=0}^{n} y_k \ell_k(x)$$

$p_n(x_i) = y_i$

$\Rightarrow \ell_j(x_i) = \delta_{ij}$

$\ell_j(x)$ is a polynomial that depends on the nodes $x_0, x_1, \dots, x_n$

but not on the ordinates $y_0, y_1, \dots, y_n$

Note that these properties define $\ell_j(x)$ uniquely given $x_0, x_1, \dots, x_n$!

we want to fit a polynomial to

Fix i and set $y_j = \delta_{ij}$. Then, $(x_0, 0), (x_1, 0) \cdots (x_i, 1), \cdots (x_n, 0)$

$$\delta_{ij} = y_j = p_n(x_j) = \sum_{k=0}^{n} y_k \ell_k(x_j) = \sum_{k=0}^{n} \delta_{ik}\ell_k(x_j) = \ell_i(x_j)$$

that is ℓ_i is a polynomial that passes through

$$(x_0, 0), (x_1, 0), \cdots, (x_i, 1), \cdots, (x_n, 0)$$

Since $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are roots of ℓ_i,

$$\ell_i(x) = c(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)$$

$$1 = \ell_i(x_i) = c(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)$$

$$= c\prod_{\substack{0\leq j\leq n\\ j\neq i}}(x_i - x_j)$$

$$\Rightarrow c = \prod_{\substack{0\leq j\leq n\\ j\neq i}}(x_i - x_j)^{-1}$$

$$\Rightarrow \ell_i(x) = \prod_{\substack{0\leq j\leq n\\ j\neq i}}\frac{(x-x_j)}{(x_i-x_j)}$$

"the cardinal functions" for the set of nodes $x_0, \dots, x_n$.

$$p(x) = \sum_{k=0}^{n} y_k\ell_k(x) = \sum_{k=0}^{n} y_k \prod_{\substack{k=0\\ k\neq i}}^{n}\frac{x-x_k}{x_i-x_k}$$

is the Lagrange form of the interpolating polynomials.

Example Find the lowest degree polynomial passing through

$(2,2), (3,8), (5,0)$

$$\ell_0(x) = \frac{(x-3)(x-5)}{(2-3)(2-5)} = \frac{1}{3}(x-3)(x-5)$$

$$\ell_1(x) = \frac{(x-2)(x-5)}{(3-2)(3-5)} = -\frac{1}{2}(x-2)(x-5)$$

$$\ell_2(x) = \frac{(x-2)(x-3)}{(5-2)(5-3)} = \frac{1}{6}(x-2)(x-3)$$

$$p_3(x) = y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x) = \frac{2}{3}(x-3)(x-5) - \frac{3}{2}(x-2)(x-5)$$

Example Find the lowest degree polynomial passing through

x	x_0	x_1
y	y_0	y_1

$$p(x) = y_0\left(\frac{x-x_1}{x_0-x_1}\right) + y_1\left(\frac{x-x_0}{x_1-x_1}\right)$$

If we want polynomial of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

then $p(x_i) = y_i \quad 0\leq i\leq n$ gives us

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n\end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n\end{pmatrix}$$

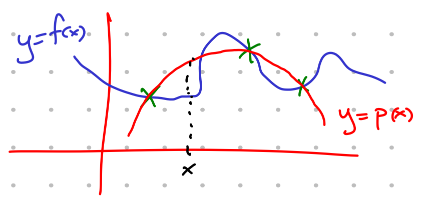
"Vandermonde Matrix" $(n+1)\times(n+1)$

What does Thm 1 say about this system?

The system has a unique solution.

Therefore, the Vandermonde matrix is non-singular.

In practice the above system is ill-conditioned so we don't use it to solve for coefficients.



The Error in Polynomial Interpolation.

Theorem 2 Let $f \in C^{n+1}[a,b]$, $x_0, \cdots x_n \in [a,b]$ be distinct points and p be the polynomial of degree at most n s.t.

$$p(x_i) = f(x_i) \quad 0\leq i\leq n.$$

Then $\forall\ x\in[a,b] \quad \exists\, y \in (a,b)$ s.t.

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(y) \prod_{i=0}^{n} (x - x_i)$$

Pf Obvious for $x = x_k$ for some node x_k.

Thus, we can assume $x \neq x_k$ for any node n_k. (x is fixed)

Set $w(t) = \prod_{i=0}^{n} (t - x_i)$ and $g = f - p - \lambda w$

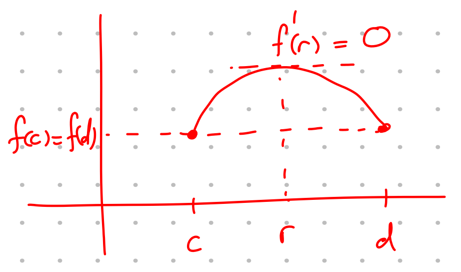
where $\lambda \in \mathbb{R}$ is chosen so that $g(x) = 0$,

that is, $\lambda = \dfrac{f(x) - p(x)}{w(x)}$

Note $g \in C^{n+1}[a,b]$ and $g(x) = g(x_0) = g(x_1) = \cdots = g(x_n) = 0$.

n+2 zeros

$\Rightarrow$ g' has at least $n+1$ distinct zeros by Rolle's theorem (special case of the MVT).



Same reasoning $\Rightarrow$

g'' has at least n distinct zeros

$\vdots$

$g^{(n+1)}$ has at least 1 zero, say $y \in (a,b)$ $\left(g^{(n+1)}(y) = 0 \right)$

Since $g^{(n+1)} = f^{(n+1)} - p^{(n+1)} - \lambda w^{(n+1)}$

$= f^{(n+1)} - (n+1)!\, \lambda$,

$w = (x - x_0)(x - x_1) \cdots (x - x_n)$

$x^{n+1} + \cdots$

$$0 = g^{(n+1)}(y) = f^{(n+1)}(y) - (n+1)!\,\lambda = f^{(n+1)}(y) - (n+1)!\, \frac{f(x) - p(x)}{w(x)}$$

$$\Rightarrow \quad f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(y)\, w(x)$$

$p(x) = x^2 + x - 1$

$p'(x) = 2x + 1$

$p''(x) = 2$ $\quad p'''(x) = 0$

Example If $f(x) = \sin x$ is approximated by a polynomial of degree 9 that interpolates f at 10 points in $[0,1]$, how large is the error on this interval?

$f^{(10)}(y) = -\sin(y) \quad \Rightarrow \quad |f^{(10)}(y)| \leq 1$

$|w(x)| = \prod_{i=0}^{9} |x - x_i| \leq 1$

So, $\forall x \in [0,1]$ $\quad |\sin x - p(x)| \leq \dfrac{1}{10!} < 2.8 \cdot 10^{-7}$

Chebyshev Polynomials

Theorem 2 can be optimized by choosing special nodes!

Chebyshev Polynomials (of the first kind) are defined by

$T_0(x) = 1 \qquad T_1(x) = x$

$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \qquad (n \geq 1)$

$\Rightarrow T_2(x) = 2x^2 - 1$

$T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$

$T_4(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$

all powers of 2.

Theorem 3

For $x \in [-1,1]$, $\quad T_n(x) = \cos(n \cos^{-1} x) \qquad (n \geq 0)$

Pf

Recall $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \cos((n+1)\theta) = \cos\theta \cos n\theta - \sin\theta \sin(n\theta)$

$\cos((n-1)\theta) = \cos\theta \cos n\theta + \sin\theta \sin(n\theta)$

$\left(\sin(-\theta) = -\sin\theta \right)$

$\Rightarrow \cos(n+1)\theta = 2\cos\theta \cos n\theta - \cos(n-1)\theta \qquad (*)$

If $\theta = \cos^{-1} x$ and $f_n(x) = \cos n\theta$, then

$\cos\theta = x$

$f_0(x) = \cos 0 = 1 \qquad f_1(x) = \cos(\cos^{-1} x) = x$

and $(*)$ gives us

$f_{n+1}(x) = 2x f_n(x) - f_{n-1}(x) \quad \Rightarrow \quad f_n(x) = T_n(x)$.

Since $T_n(x) = \cos(n \cos^{-1} x)$, $(-1 \leq x \leq 1)$

1) $|T_n(x)| \leq 1$

2) $T_n\left(\cos \frac{j}{n}\pi\right) = (-1)^j$

$\cos\left(n\left(\frac{j\pi}{\cancel{n}}\right)\right)$

3) $T_n\left(\cos\left(\frac{2j-1}{2n}\pi\right)\right) = 0$

Def^n $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is called a monic polynomial if $a_n = 1$.

Note that $T_n(x)/2^{n-1} = 2^{1-n} T_n(x)$ is a monic polynomial.

Theorem 4 If p is a monic polynomial of degree n, then

$$\|p\|_\infty = \max_{-1\le x\le 1} |p(x)| \geq 2^{1-n}$$

Pf Suppose $|p(x)| < 2^{1-n}$ $\forall x \in [-1,1]$.

Let $q = 2^{1-n} T_n$ and $x_i = \cos\left(\frac{i\pi}{n}\right)$.

$$(-1)^i p(x_i) \le |p(x_i)| < 2^{1-n} = (-1)^i q(x_i) = (-1)^i (2^{1-n}) \underbrace{T_n(x_i)}_{=(-1)^i} \quad = 2^{1-n}$$

$\Rightarrow$ $(-1)^i [q(x_i) - p(x_i)] > 0$ $\quad 0 \le i \le n$

$\Rightarrow$ $q-p$ have at least n roots on $(-1,1)$ (why?) by IVT

However, $q-p$ is of degree at most $n-1$ (why?) they are monics

This is a contradiction! (why?) $q-p=0 \Rightarrow p=q$

but $\max_{-1\le x\le 1} |q(x)| \geq 2^{1-n}$

Choosing the nodes

Consider $f \in C^{(n+1)}[-1,1]$, by Thm 2,

$$\max_{|x|\le 1} |f(x) - p(x)| \le \frac{1}{(n+1)!} \max_{|x|\le 1} |f^{(n+1)}(x)| \max_{|x|\le 1} \left|\prod_{i=0}^{n} (x - x_i)\right|$$

Thm 4 $\Rightarrow$ $\max_{|x|\le 1} \left|\prod_{i=0}^{n} (x - x_i)\right| \geq 2^{1-n}$

Equality when $\prod_{i=0}^{n} (x - x_i) = 2^{1-(n+1)} T_{n+1}(x) = 2^{-n} T_{n+1}(x)$

$\Rightarrow$ x_i are roots of $T_{n+1}(x)$, that is

$$x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right) \quad 0 \le i \le n$$

Theorem 5

If $x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right)$ $\quad 0 \le i \le n$ then

$$|f(x) - p(x)| \le \frac{1}{2^n (n+1)!} \max_{|y|\le 1} |f^{(n+1)}(y)|$$