<u>Last time</u>: <u>Thm</u>: Singular Value Decomposition (SVD)

Let A be an arbitrary $m\times n$ matrix (with $\mathbb{C}$ coefficients).
Then $\exists$ $m\times m$ P, $n\times n$ Q, $m\times n$ D s.t.
$A = PDQ$, $P^*P = I$, $Q^*Q = I$ (unitary) and D is diagonal.

First, we find eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2 \geq \sigma_{r+1}^2 = 0 \geq \cdots \geq \sigma_n^2$
and corresponding (o.n.) eigenvectors $u_1, u_2, \ldots, u_n$ of A^*A.

Then
$$Q = \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix}_{n\times n} \quad \text{and} \quad D = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}_{m\times n}$$

Define $v_i = \sigma_i^{-1} A u_i$ $(1 \le i \le r)$ and extend to an o.n. basis $\{v_1 \cdots v_m\}$ for $\mathbb{C}^m$.

Then
$$P = [v_1 \ \cdots \ v_m]$$

$\sigma_1, \ldots, \sigma_n$ are called the <u>singular values</u> of A.

<u>Example</u> Find an SVD of $A = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$A^*A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 49 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \sigma_1 = 7 \quad \sigma_2 = 3 \quad \sigma_3 = 0 = \sigma_4$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ since rows of Q are eigenvectors of A^*A.

$A^*Ae_i = (A^*A)_{ii} e_i$ where $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

$$D = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$v_1 = \sigma_1^{-1} A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 7^{-1}\begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = \sigma_2^{-1} A \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$v_i = \sigma_i^{-1} A u_i$ $\quad 1 \le i \le \underset{\overset{\shortparallel}{2}}{r}$

then we extend $\{v_1, \ldots v_r\}$ to o.n. basis $\{v_1, \ldots v_r, \ldots v_m\}$

then we can choose $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly $A = PDQ$

Another SVD for A is
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

<u>Example</u> Find a SVD for $A = \begin{pmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$A^*A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\sigma_1^2 = 4 \geq \sigma_2^2 = 1 \geq \sigma_3^2 = 1$

set $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = 0$

We can choose $\underset{\overset{\shortparallel}{u_1}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{\overset{\shortparallel}{u_2}}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}, \underset{\overset{\shortparallel}{u_3}}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ as eigenvectors.

So $Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_1^* \\ u_2^* \\ u_3^* \end{pmatrix}$ $\qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$v_1 = 2^{-1} A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.8 \\ 0.6 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = 1^{-1} A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \\ 0 \\ 0 \end{pmatrix}$$

We can choose $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Thus,
$$A = \underbrace{\begin{pmatrix} -0.8 & 0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P}\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{Q}$$

<u>Pseudo inverses</u>

Let $\underset{m\times n}{D} = \begin{pmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & 0 & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$. We define the <u>pseudoinverse</u> D^+ of D to be the $n\times m$ matrix

$$D^+ = \begin{pmatrix} \sigma_1^{-1} & & & & 0 \\ & \sigma_2^{-1} & & & \\ & & \ddots & & \\ & & & \sigma_r^{-1} & \\ & 0 & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

For $A \in \mathbb{C}^{m\times n}$, first find a SVD $A = PDQ$. Then its <u>pseudoinverse</u> is $A^+ = Q^* D^+ P^* \in \mathbb{C}^{n\times m}$

<u>Remark</u>: Even though the SVD is not unique, the pseudoinverse is unique! (If A^{-1} exists, $A^{-1} = A^+$)

<u>Example</u> Let
$$A = \begin{pmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then
$$A^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^*\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^+\begin{pmatrix} -0.8 & 0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.8 & 0.6 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -0.4 & 0.3 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0.3 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \end{pmatrix}$$

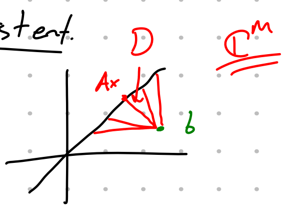
Inconsistent and Undetermined Systems

(Say A is invertible, then $Ax=b \Rightarrow x = A^{-1}b$

Consider $Ax = b$ for an $m \times n$ matrix A.

Then, the minimal solution of this system is the least squares solution that has the least Euclidean norm if the system is inconsistent and it is the exact solution with the least Euclidean norm if the system is consistent.

$A \in \mathbb{C}^{m \times n} \Rightarrow L_A : \mathbb{C}^n \to \mathbb{C}^m \quad L_A(x) = Ax$



Alternative definition: let $D = \inf_{x \in \mathbb{C}^n} \|Ax - b\|_2$

Then the minimal solution of $Ax=b$ is the (unique) element $x_0 \in \mathbb{C}^n$ satisfying $\|x_0\|_2 = \inf\{ \|x\|_2 \mid x \in \mathbb{C}^n, \|Ax-b\|_2 = D\}$

Theorem The minimal solution of $Ax=b$ is given by $x = A^+ b$.

Example Find the minimal solution of

$$\begin{aligned} 0x - 1.6y + 0.6z &= 5 \\ 0x + 1.2y + 0.8z &= 7 \\ 0x + 0y + 0z &= 3 \\ 0x + 0y + 0z &= -2 \end{aligned} \quad \rightarrow \quad \underbrace{\begin{pmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^+ \begin{pmatrix} 5 \\ 7 \\ 3 \\ 2 \end{pmatrix}$$

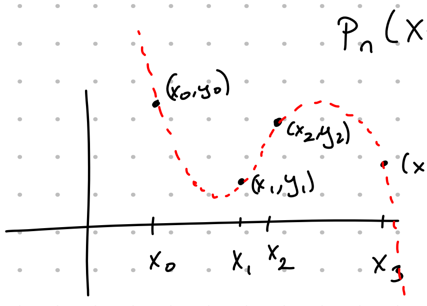
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0.3 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.1 \\ 8.6 \end{pmatrix}$$

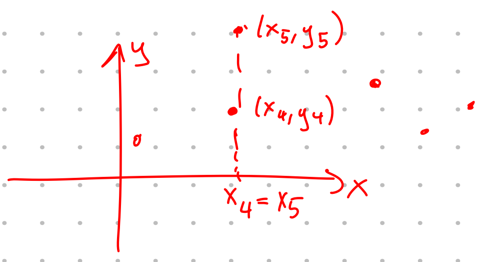
Chp 6 Approximating Functions

6.1 Polynomial Interpolation

Theorem If $x_0, x_1, \dots, x_n$ are distinct real numbers, then for arbitrary values $y_0, y_1, \dots, y_n$, $\exists!$ polynomial p_n of degree at most n such that

$$p_n(x_i) = y_i \qquad (0 \le i \le n)$$





Pf We prove uniqueness first.

Say p_n and q_n are two polynomials satisfying the conditions of the theorem. Then,

$$p_n(x_i) = y_i = q(x_i) \quad \Rightarrow \quad (p-q)(x_i) = 0 \qquad 0 \le i \le n$$

Note that $p-q$ is a polynomial of degree at most n but it has $n+1$ zeros. Thus, it must be the constant 0 polynomial.

Next existence: We use induction on n.

For $n=0$, set $p(x) = y_0$ ($\deg p = 0$) ✓

Assume that the statement is true for $n = k-1$.

Given $x_0, x_1, \dots, x_{k-1}, x_k$, $y_0, \dots, y_{k-1}, y_k$

by induction hypothesis, $\exists\, p_{k-1}$ of degree $\le k-1$ s.t.

$p_{k-1}(x_i) = y_i \quad (0 \le i \le k-1)$

Define $p_k(x) = p_{k-1}(x) + c(x-x_0)(x-x_1)\cdots(x-x_{k-1})$ for some $c \in \mathbb{R}$ (to be determined below)

Clearly $\deg(p_k) \le k$,

$$(x_i - x_i)$$

$$p_k(x_i) = p_{k-1}(x_i) = y_i \qquad \text{for } i \le k-1.$$

and $p_k(x_k) = p_{k-1}(x_k) + c(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1}) = y_k$ ✳

Thus, it is clear that $\exists c \in \mathbb{R}$ s.t. $p_k(x_k) = y_k$.

Note that the proof gives us

$p_0(x) = c_0$

$p_1(x) = c_0 + c_1(x - x_0)$

$p_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$

$\vdots$

$p_n(x) = c_0 + c_1(x-x_0) + \dots + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$

(for some appropriate choices of c_i)

These polynomials are called the interpolation polynomials in Newton's form.

To evaluate $P_n(x)$, we will use Horner's algorithm

Recall (Horner's Algorithm) $\quad d_0 = (x - x_0) \quad d_1 = (x - x_1) \cdots$

If $u = c_0 + c_1 d_0 + c_2 d_0 d_1 + \cdots + c_k d_0 d_1 \cdots d_{k-1}$

then $u = (\cdots (\underbrace{(\underbrace{(c_k) d_{k-1} + c_{k-1})}\, d_{k-2} + c_{k-2})}\, d_{k-3} + \cdots + c_1) d_0 + c_0$.

We can compute u by the following series of operations

$u_k \leftarrow c_k$

$u_{k-1} \leftarrow \underline{u_k d_{k-1} + c_{k-1}}$

$u_{k-2} \leftarrow \underbrace{u_{k-1} d_{k-2} + c_{k-2}}$

$\vdots$

$u = u_0 \leftarrow u_1 d_0 + c_0$

or more compactly by,

```
u ← c_k
for i = k-1 to 0 step -1 do
    u ← u d_i + c_i
end do
```

for our polynomial, $d_i = (x - x_i)$ $\quad \left[P_n(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x-x_0)\cdots(x - x_{n-1}) \right]$

Thus, we can compute $u = p_k(t)$ (for a fixed t) using

```
u ← c_k
for i = k-1 to 0 step -1 do
    u ← (t - x_i) u + c_i
end do
```

$d_i = (t - x_i)$

Note that $c_k = \dfrac{y_k - P_{k-1}(x_k)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})}$ using $\otimes$

which we can compute by using similar algorithms but we will talk about a faster way to compute them later.

Example Find the lowest degree polynomial that passes through $(2,2)$, $(3,8)$, and $(5,0)$.

i.e.

i	0	1	2
x_i	2	3	5
y_i	2	8	0

$P_0(x) = c_0 = y_0 = 2$

$P_1(x) = P_0(x) + c_1(x - x_0) = 2 + c_1(x-2)$

$P_1(3) = 8 \Rightarrow 2 + c_1(3-2) = 8 \Rightarrow c_1 = 6$

$P_2(x) = 2 + 6(x-2) + c_2(x-2)(x-3)$

$P_2(5) = 0 \Rightarrow 2 + 6(3) + c_2(3)(2) = 0$

$6c_2 = -20 \qquad c_2 = -\frac{10}{3}$

Thus, $p(x) = P_2(x) = 2 + 6(x-2) - \frac{10}{3}(x-2)(x-3)$

$$p(x) = \left(\left(-\frac{10}{3}\right)(x-3) + 6\right)(x-2) + 2$$

$p(2) = (\qquad)(2-2) + 2 = 2$

$p(3) = \left(-\frac{10}{3}(3-3) + 6\right)(3-2) + 2 = 8$

$p(5) = \left(-\frac{10}{3}(5-3) + 6\right)(5-2) + 2 = \left(-\frac{20}{3} + 6\right)3 + 2 = 0$