Common mistakes so far (not all relevant to the midterm exam)

Order of quantifiers: (Example)

$f:(a,b)\to\mathbb{R}$ satisfies P1 if $\exists \lambda<1$ s.t. $\forall x\in(a,b)$ $f(x)\le\lambda$

$\not\parallel$ P2 if $\forall x\in(a,b)$ $\exists\lambda<1$ s.t. $f(x)\le\lambda$

(implicitly depends on x)

equivalently, we can write

P2: if $\forall x\in\mathbb{R}$, $\exists\lambda_x<1$ s.t. $\boxed{f(x)\le\lambda_x}$ to show the dependence on x more explicitly.

a) Prove that if f satisfies P1 then it satisfies P2.

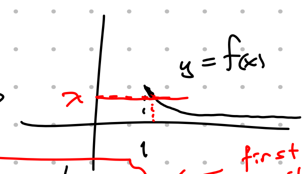
b) Find examples of f that satisfy P2 but not P1.

We want to show that f satisfies P2: Given $x\in\mathbb{R}$ (we need to find $\lambda_x<1$ s.t. $f(x)\le\lambda_x$). Since f satisfies P1, $\exists\lambda<1$ s.t. $f(x)\le\lambda$ $\forall x\in\mathbb{R}$. All we have to do is set $\underset{\text{needed in P2}}{\lambda_x} = \underset{\text{comes from P1}}{\lambda}$ thus $f(x)\le\lambda=\lambda_x$ ⓐ ✓

if (this color is green) then $x^2>-1$ for $x\in\mathbb{R}$

ⓑ $\frac{1}{x}$

on $(1,\infty)$ →



for any $x>1$, $\frac{1}{x}<1$. $\boxed{\text{Set } \lambda_x=\frac{1}{x}<1}$ ← first found a suitable λ_x then proved P2

then $\forall x>1$ $\lambda_x=\frac{1}{x}$ so $f(x)=\frac{1}{x}\le\lambda_x$ P2 ✓

P1 is not satisfied since $\forall\ 0<\lambda<1$, $\frac{1}{\lambda}>1 \Rightarrow \exists x$ s.t. $\frac{1}{\lambda}>x>1$, $x\in(1,\infty)$ but $f(x)=\frac{1}{x}>\lambda$

$\lambda<\frac{1}{x}$

Technicalities of ε-δ proofs

$\lim_{x\to a} f(x)=L$ if

$$\left(\forall\varepsilon>0\ \exists\delta>0 \text{ s.t. } \forall x\in\mathbb{R} \text{ (that satisfies) } 0\ne|x-a|<\delta,\ |f(x)-L|<\varepsilon\right) \text{ is true}$$

Note that δ may depend on ε but not on x.

More generally, The limit $\lim_{x\to a} f(x)$ exists if

$$\exists L\in\mathbb{R} \text{ s.t. } \left(\begin{array}{l}\forall\varepsilon>0\ \exists\delta>0 \text{ s.t.}\\ \forall x\in\mathbb{R} \text{ (that satisfies) } 0\ne|x-a|<\delta,\ |f(x)-L|<\varepsilon\end{array}\right) \text{ is true}$$

We negate it as follows:

The limit $\lim_{x\to a} f(x)$ does not exist if

$$\forall L\in\mathbb{R} \left(\begin{array}{l}\forall\varepsilon>0\ \exists\delta>0 \text{ s.t.}\\ \forall x\in\mathbb{R} \text{ (that satisfies) } 0\ne|x-a|<\delta,\ |f(x)-L|<\varepsilon\end{array}\right) \text{ is false}$$

Example Prove that $\lim_{x\to 1} 2x+5=7$. if P1 then P2

We cannot (in general) instantly start proving the formal definition

$$\left(\forall\varepsilon>0\ \exists\delta>0 \text{ s.t. } \forall x\in\mathbb{R} \text{ (that satisfies) } |x-a|<\delta,\ |f(x)-L|<\varepsilon\right)$$

because we don't know what δ works. Instead we first do some work to figure out a suitable δ and then prove the formal statement using a δ that works!

figure out what δ works

We want

$|2x+5-7|<\varepsilon$ whenever $0\ne|x-1|<\delta$ reduce it to this

$\parallel$

$|2x-2|=2|x-1|<\varepsilon$

so $|x-1|<\frac{\varepsilon}{2}$. Thus, it seems like $\delta=\frac{\varepsilon}{2}$ should work.

actual proof

Given $\varepsilon>0$, let $\delta=\frac{\varepsilon}{2}>0$. Then

$|x-1|<\delta=\frac{\varepsilon}{2} \Rightarrow 2|x-1|<\varepsilon$

$\Rightarrow |2x-2|<\varepsilon$

$|2x+5-7|<\varepsilon$

i.e. $|f(x)-L|<\varepsilon$ which was to be shown.

When computing $fl(x)$

$u=2^{-23}$ $e=\frac{3}{4}u$

$1+e==1$ but $1-e\ne1$

First, you need to find the binary representation of x.

e.g. $x=\frac{5}{32}=(0.00101)_2$ (fixed point)

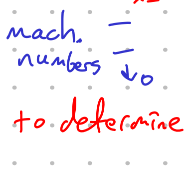
Then write x in the form $x=(1.f)_2\times2^m$. (floating point)

e.g. $\frac{5}{32}=(1.01)_2\times2^{-3}$

Then we truncate the bits after the 23^{rd} digit to the right of the binary point in floating point representation, to get x_-.

To get x_+, we add 1 to the 23^{rd} digit, that is,

$$x_+=x_-+2^{-23}\cdot2^m = x_-+2^{-26}$$



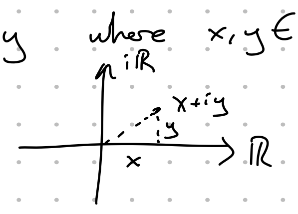
Then check if x is closer to x_+ or x_- to determine which one is $fl(x)$.

Chp 5 Selected Topics in Numerical Linear Algebra

Review of Basic Concepts

$\mathbb{C}$: Complex numbers $z = x + iy$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$

that is $i^2 = -1$



conjugate: $\bar{z} = x - iy$

modulus : $|z| = \sqrt{x^2 + y^2}$

Notice that $\bar{\bar{z}} = z$ and $z\bar{z} = |z|^2$

Fundamental Theorem of Algebra

$x^2 + 1$: no roots over $\mathbb{R}$.

Over $\mathbb{C}$, every non-constant polynomial has at least one root.

If f is polynomial of degree n, then it is a product of n linear terms

$f(z) = \lambda(z - z_1)(z - z_2) \cdots (z - z_n)$

The (standard) inner product and the Euclidean norm on $\mathbb{C}^n$ is defined by

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \bar{y}_j = x^T \bar{y}$$

$$\|x\|_2 = \sqrt{\langle x, x \rangle}$$ ← real valued quantity.

Notice that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$

↑ $\mathbb{C}$-anti linear.

and $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$\langle x, y \rangle$ is $\mathbb{C}$-linear in x and $\mathbb{C}$-anti linear in y.

If A is a complex $n \times k$ matrix then A^* is the (complex) $k \times n$ matrix

$$A^* = \bar{A}^T \quad \text{that is} \quad (A^*)_{jk} = \bar{A}_{kj}$$

Note that $\overset{\text{scalar}}{\langle x, y \rangle} = x^T \bar{y} = (x^T \bar{y})^T = \bar{y}^T x^T{}^T = y^* x$

and

$$x^* x = \langle x, x \rangle = \|x\|_2^2 = \sum_{j=1}^{n} x_j \bar{x}_j = \sum_{j=1}^{n} |x_j|^2$$