Condition Number

Consider $Ax=b$ where A is $n\times n$ and invertible.

Example

Say A^{-1} is perturbed (e.g. due to roundoff errors) to obtain a new matrix B.

Then the solution $x = A^{-1}b$ is pertubed to $\tilde{x} = Bb$. $\boxed{\|Ax\| \le \|A\|\,\|x\|}$

Abs. Error $= \|x-\tilde{x}\| = \|x - Bb\| = \|x - BAx\| = \|(I-BA)x\| \le \|I-BA\|\,\|x\|$

Thus, Relative Error $= \dfrac{\|x-\tilde{x}\|}{\|x\|} \le \|I-BA\|$ $\quad B \simeq A^{-1}$

Example

Suppose b is pertubed to $\tilde{b}$, $\underbrace{Ax=b}_{\text{exact}}$ and $\underbrace{A\tilde{x}=\tilde{b}}_{\text{approximate}}$

Then

Abs. Error $= \|x-\tilde{x}\| = \|A^{-1}b - A^{-1}\tilde{b}\| = \|A^{-1}(b-\tilde{b})\| \le \|A^{-1}\|\,\|b-\tilde{b}\|$

$$\|x-\tilde{x}\| \le \|A^{-1}\|\,\|b\|\,\frac{\|b-\tilde{b}\|}{\|b\|} = \|A^{-1}\|\,\|Ax\|\,\frac{\|b-\tilde{b}\|}{\|b\|}$$

$$\le \|A^{-1}\|\,\|A\|\,\|x\|\,\frac{\|b-\tilde{b}\|}{\|b\|}$$

Thus, Relative Error $= \dfrac{\|x-\tilde{x}\|}{\|x\|} \le K(A)\dfrac{\|b-\tilde{b}\|}{\|b\|}$ where $\boxed{K(A) = \|A^{-1}\|\,\|A\|}$ is called the condition number.

Note that $\boxed{I = A^{-1}A}$

$$\Rightarrow \quad 1 = \|I\| = \|A^{-1}A\| \le \|A^{-1}\|\,\|A\| = K(A).$$

Example

let $A = \begin{pmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{pmatrix}$ $\quad A^{-1} = \varepsilon^{-2}\begin{pmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{pmatrix}$ $\quad (\varepsilon > 0)$

$1 + |-1-\varepsilon| = 2+\varepsilon$

Then $\|A\|_\infty = \overset{1+1+\varepsilon}{2+\varepsilon}$ $\quad \|A^{-1}\|_\infty = \varepsilon^{-2}(2+\varepsilon)$

Hence $K(A) \overset{\|A\|_\infty \|A^{-1}\|_\infty}{=} \left(\dfrac{2+\varepsilon}{\varepsilon}\right)^2 > \dfrac{4}{\varepsilon^2}$. If $\varepsilon \le 10^{-2}$ then

$$K(A) \ge 4\cdot 10^4 = 40000$$

Thus a small error in b may induce a perturbation of x that is 40000 times larger.

let $\tilde{x}$ be the numerical approximation to $Ax=b$. let $r = b - A\tilde{x}$ $\left(Ax - A\tilde{x}\right)$ called the residual vector. Set $e = x - \tilde{x}$ to be the error vector.

Note

$$Ae = Ax - A\tilde{x} = b - A\tilde{x} = r$$

Set $\tilde{b} = A\tilde{x}$ then $\tilde{b} = b - r$ is the perturbed right hand side vector.

Now, $\boxed{\dfrac{\|x-\tilde{x}\|}{\|x\|} = \dfrac{\|e\|}{\|x\|}}$ and $\boxed{\dfrac{\|b-\tilde{b}\|}{\|b\|} = \dfrac{\|r\|}{\|b\|}}$

Thm When solving $Ax=b$ (for invertible A) we have

$$\frac{1}{K(A)}\frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le K(A)\frac{\|r\|}{\|b\|}$$

?

Note that we have already proved the inequality on the right above

The inequality on the left can be proved as follows: $\boxed{\|Mv\| \le \|M\|\,\|v\|}$

$$\|r\|\,\|x\| = \|Ae\|\,\|A^{-1}b\| \overset{2\times}{\le} \|A\|\,\|e\|\,\|A^{-1}\|\,\|b\| = K(A)\,\|e\|\,\|b\|$$

thus $\quad \dfrac{1}{K(A)}\dfrac{\|r\|}{\|b\|} \le \dfrac{\|e\|}{\|x\|}$.

4.5 Neumann Series and Iterative Refinement

Given a normed vectorspace $(V, \|\cdot\|)$ and a sequence $\{v^{(n)}\}$ in V, we say that $\{v^{(n)}\}$ converges to v if

$$\lim_{n\to\infty} \|v^{(n)} - v\| = 0$$

e.g. $v^{(n)} = \begin{pmatrix} 3 - 1/n \\ -2 + \frac{1}{\sqrt{n}} \\ 1 \\ e^{-n} \end{pmatrix} \in \mathbb{R}^4$ let $v = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

Then $\|v^{(n)} - v\|_\infty = \left\|\begin{pmatrix} -1/n \\ 1/\sqrt{n} \\ 0 \\ e^{-n} \end{pmatrix}\right\|_\infty = \dfrac{1}{\sqrt{n}} \to 0$ as $n\to\infty$

In fact, for finite dimensional V, the notion of convergence is independent of the norm. (If $v^{(n)} \to v$ using norm $\|\cdot\|$, then $v^{(n)} \to v$ using any norm $\|\cdot\|'$.)

Cauchy criterion

$a_n \in \mathbb{R}$ $\quad \{a_n\}$ is Cauchy if $\forall \varepsilon > 0$ $\exists N$ s.t. $|a_i - a_j| < \varepsilon$ $\forall\, i,j \ge N$

$\{v^{(n)}\}$ converges (in finite dimensional V over $\mathbb{R}$ or $\mathbb{C}$) iff $\displaystyle\lim_{n\to\infty}\sup_{i,j\ge n} \|v^{(i)} - v^{(j)}\| = 0$.

Thm (Theorem on Neumann Series)

If A is $n\times n$ and $\boxed{\|A\| < 1}$, then $I - A$ is invertible and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Recall geometric series $|r|<1$ then $\dfrac{1}{1-r} = \displaystyle\sum_{k=0}^{\infty} r^k$

Pf

Assume on the contrary that $I-A$ is not invertible.

Then $\exists\, x \in \mathbb{R}^n$ s.t. $\|x\| = 1$ and $(I-A)x = 0 \Rightarrow Ax = x$

Thus, $1 = \|x\| = \|Ax\| \le \|A\|\,\|x\| = \|A\|$ which is a contradiction.

Next we want to show $\sum_{k=0}^{m} A^k \to (I-A)^{-1}$ or equivalently

$$(I-A)\sum_{k=0}^{m} A^k \to I.$$

Note that $(I-A)\sum_{k=0}^{m} A^k = \sum_{k=0}^{m} A^k - A^{k+1} = A^0 - \cancel{A^1} + \cancel{A^1} - \cancel{A^2} + \cdots + \cancel{A^m} - A^{m+1}$

$$= A^0 - A^{m+1} = I - A^{m+1}$$

Note that $\|A^{m+1}\| \le \|A\|\,\|A^m\| \le \|A\|\,\|A\|\,\|A^{m-1}\| \le \cdots \le \|A\|^{m+1}$ (*)

Since $\|A\|<1$, $\|A^{m+1}\| \le \|A\|^{m+1} \to 0$

$\Rightarrow A^{m+1} \to 0$ which was to be shown.

Remark: by the prev. thm and triangle ineq.; geometric series with $r=\|A\|$; by (*)

$$\|(I-A)^{-1}\| \le \sum_{k=0}^{\infty} \|A^k\| \le \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1-\|A\|}$$

Example Find the inverse of $B = \begin{pmatrix} 0.9 & -0.3 \\ 0.3 & 1.1 \end{pmatrix}$ using Neumann series.

Set $A = I - B = \begin{pmatrix} 0.1 & 0.3 \\ -0.3 & -0.1 \end{pmatrix}$ Clearly $\|A\|_\infty = 0.4 < 1$

Thus, $B = I - A$ is invertible and $B^{-1} = \sum_{k=0}^{\infty} A^k$

Set (Partial sums) $B_m = \sum_{k=0}^{m} A^k$ ($B_m \to B^{-1}$ as $m\to\infty$)

Then $B_0 = I$. $B_1 = I + A = I + AB_0$

$B_2 = I + A + A^2 = I + AB_1$ and so on. $B_{n+1} = I + AB_n$

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1.1 & 0.3 \\ -0.3 & 0.9 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1.02 & 0.3 \\ -0.3 & 0.82 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 1.012 & 0.276 \\ -0.276 & 0.828 \end{pmatrix} \quad \cdots$$

$$B^{-1} \approx \begin{pmatrix} 1.0185 & 0.27778 \\ -0.27778 & 0.83333 \end{pmatrix}$$

Thm2 If A and B are $n\times n$ matrices s.t. $\|I-AB\|<1$ (r), then A and B are invertible. Furthermore,

$$\boxed{A^{-1} = B\sum_{k=0}^{\infty}(I-AB)^k} \ (*) \quad \text{and} \quad B^{-1} = \sum_{k=0}^{\infty}(I-AB)^k A \ (\circledast)$$

Pf By the prev. thm, $I - (I-AB) = AB$ is invertible and

$$(AB)^{-1} = \sum_{k=0}^{\infty}(I-AB)^k \qquad \text{Since } (AB)^{-1} = B^{-1}A^{-1},$$

we have $B^{-1}A^{-1} = \sum_{k=0}^{\infty}(I-AB)^k$. The desired equalities are easy to obtain from here.

Multiply from left by B to get (*)
Multiply from right by A to get ($\circledast$)

Iterative Refinement

Say $x^{(0)}$ is an approximate sol.^{n} to $Ax=b$

(for example the solution obtained numerically using Gaussian elimination)

Q) How to improve the accuracy of the solution?

A) Define

1) $r^{(k)} = b - Ax^{(k)}$ the residual vector corresponding to $x^{(k)}$

2) $Ae^{(k)} = r^{(k)}$ the error vector

3) $x^{(k+1)} = x^{(k)} + e^{(k)}$ for $k \ge 0$

Then $x^{(k)} \to x$: the exact solution as $k \to 0$.

Here's why:

Say B is an approximate inverse to A. More precisely, B is a matrix that satisfies $\|I-BA\|<1$. $Ax=b$

We have $x^{(0)} = Bb$ then combining steps 1, 2, 3 we have

$$x^{(k+1)} = x^{(k)} + B(b - Ax^{(k)})$$

Thus $\boxed{x^{(k+1)} - x} = x^{(k)} - x + B(b - Ax^{(k)})$

$$= x^{(k)} - x + B(Ax - Ax^{(k)})$$

$$\boxed{= (I-BA)(x^{(k)} - x)}$$

Therefore, $\boxed{\|x^{(k+1)} - x\|} = \|(I-BA)(x^{(k)}-x)\|$

$$\le \|I-BA\|\,\|x^{(k)}-x\|$$

$$\le \|I-BA\|^2\,\|x^{(k-1)}-x\|$$

$$\vdots$$

$$\boxed{\le \|I-BA\|^k\,\|x^{(0)}-x\|}$$

Since $\|I-BA\|<1$, $\|x^{(k)}-x\| \to 0$ as $k\to\infty$.

Note that Thm2 gives us a way to compute A^{-1} very accurately starting with an initial approximation, that is, B with $\|I-AB\|<1$

Then $A^{-1} = B\sum_{k=0}^{\infty}(I-AB)^k$.

That is, $A_0 = B$ and $\boxed{A_{n+1} = B + A_n(I-AB)}$

gives $A_n \to A^{-1}$ as $n\to\infty$

$Ax=b \Rightarrow x = A^{-1}b = B\sum_{k=0}^{\infty}(I-AB)^k\, b$