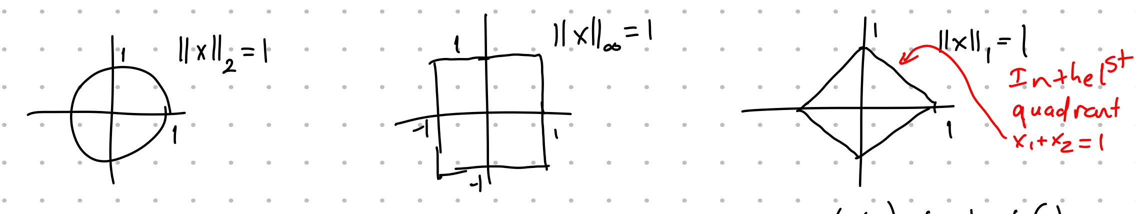
4.4 Norms and the Analysis of Errors

V: Vectorspace $\quad \|\cdot\| : V \to [0,\infty)$ is called a norm if

1) $\|x\| > 0$ for $0 \neq x \in V$
2) $\|\lambda x\| = |\lambda| \|x\|$ for $\lambda \in \mathbb{R}$, $x \in V$
3) $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in V$ (triangle inequality)

e.g. On $\mathbb{R}^n$, the standard norm is the Euclidean l_2-norm given by $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ for $x = (x_1, x_2, \cdots, x_n)$

l_∞-norm: $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$ $\quad$ l_1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$ $\quad$ $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$



Example Compute $\|x\|_i$ for $i \in \{1, 2, \infty\}$ and $x \in \left\{ \begin{pmatrix} 4 \\ 4 \\ -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\quad = v_1, \quad = v_2, \quad = v_3$

$\|v_1\|_1 = 4 \cdot |4| = 16 \qquad \|v_2\|_1 = 3|5| = 15 \qquad \|v_3\|_1 = |6| = 6$

$\|v_1\|_2 = \sqrt{4 \cdot 4^2} = 8 \qquad \|v_2\|_2 = \sqrt{3 \cdot 5^2} = 5\sqrt{3} \qquad \|v_3\|_2 = \sqrt{6^2} = 6$

$\|v_1\|_\infty = 4 \qquad \|v_2\|_\infty = 5 \qquad \|v_3\|_\infty = |6| = 6$

Matrix Norms

Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, we define

vector norms

matrix norm $\to$ $\|A\| = \sup\{\|Au\| : u \in \mathbb{R}^n, \|u\| = 1\}$ for $A \in M_{n\times n}(\mathbb{R})$

More concisely, we can write $\boxed{\|A\| = \sup_{\|u\|=1} \|Au\|}$.

This is called the matrix norm associated to the given vector norm.

Thm $\|A\| = \sup_{\|u\|=1} \|Au\|$ defines a norm.

Pf 1) We want to show $\|A\| \neq 0$ if $A \neq 0$.

If $A \neq 0$ then $\exists\, i,j$ s.t. $A_{ij} \neq 0$. let $x \in \mathbb{R}^n$ be defined by $x_k = \delta_{jk}$ so

$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$th component and $\boxed{v = \frac{x}{\|x\|}}$

Then $Av \neq 0$ and $\|A\| = \sup_{\|u\|=1} \|Au\| \geq \|Av\| \neq 0$ ✓

2) $\|\lambda A\| = |\lambda| \|A\|$

vector $\quad$ vector norm

$\|\lambda A\| = \sup_{\|u\|=1} \|\lambda Au\| = |\lambda| \sup_{\|u\|=1} \|Au\| = |\lambda| \|A\|$ ✓

scalar $\quad$ vector

3) $\boxed{\|A+B\| \leq \|A\| + \|B\|}$ triangle ineq.

vect. norm.

$\|A+B\| = \sup_{\|u\|=1} \|(A+B)u\| = \sup_{\|u\|=1} \|Au + Bu\|$

triangle ineq. for the vect. norm $\to$ $\leq \sup_{\|u\|=1} (\|Au\| + \|Bu\|) \leq \sup_{\|u\|=1} \|Au\| + \sup_{\|u\|=1} \|Bu\|$

$= \|A\| + \|B\|$ ✓

Prop. $\boxed{\|Ax\| \leq \|A\| \|x\| \text{ for } x \in \mathbb{R}^n}$ $\qquad x = Bu$

Pf If $x = 0$, we are done.

If $x \neq 0$, set $v = \frac{x}{\|x\|}$ (unit vector)

$\|Av\| \leq \sup_{\|u\|=1} \|Au\| = \|A\|$

$\|Av\| = \|A \frac{x}{\|x\|}\| = \frac{1}{\|x\|} \|Ax\| \leq \|A\|$ $\quad$ so $\|Ax\| \leq \|A\| \|x\|$

Example Find the matrix norm associated to the l_∞ norm on $\mathbb{R}^n$.

(We will denote $\|\cdot\|_\infty$ by $\|\cdot\|$)

ith component of the vector Au

$\|A\|_\infty = \|A\| = \sup_{\|u\|=1} \|Au\| = \sup_{\|u\|=1} \left(\max_{1\leq i\leq n} |(Au)_i|\right)$

$= \max_{1\leq i\leq n} \left(\sup_{\|u\|=1} |(Au)_i|\right) = \max_{1\leq i\leq n} \left(\sup_{\|u\|=1} \left|\sum_{j=1}^n A_{ij} u_j\right|\right)$

Note that $\|u\| = 1 \Rightarrow -1 \leq u_j \leq 1$ for all j

Therefore $-|A_{ij}| \leq A_{ij} u_j \leq |A_{ij}|$ and

we have $A_{ij} u_j = |A_{ij}|$ if we set $\begin{cases} u_j = 1 & \text{if } A_{ij} \geq 0 \\ u_j = -1 & \text{if } A_{ij} < 0 \end{cases}$

$\Downarrow$

$\|u\|_\infty = 1$

Thus $\sup_{\|u\|=1} \left|\sum_{j=1}^n A_{ij} u_j\right| = \sum_{j=1}^n |A_{ij}|$ and

$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |A_{ij}|$ $\quad$ for a fixed i, sum of abs. values of elements in i^{th} row.

Another important example: l_2-matrix norm (spectral norm)

$$\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 = \max_{1\leq i\leq n} |\sigma_i|$$

where σ_i are the singular values of A. More conceretly,

$\|A\|_2 = \sqrt{\rho(A^TA)}$ where $\rho(A^TA)$ is the largest eigenvalue of A^TA.

"spectral radius"

We will justify these when we talk about "Singular Value Decomposition"

Matrix norms associated to vector norms satisfy:

Thus if $\|\cdot\|$ is a norm on $\mathbb{R}^{n\times n}$ and $\|I\| \neq 1$ then it is not a matrix norm associated to a vector norm.

$$\|I\| = 1 \qquad \text{and} \qquad \|AB\| \leq \|A\|\,\|B\|$$

Pf: $\|I\| = \sup_{\|u\|=1} \|Iu\| = \sup_{\|u\|=1} \|u\| = 1.$

For the second statement, we will use (but not prove) the fact that given $A \in \mathbb{R}^{n\times n}$, $\exists\, u \in \mathbb{R}^n$ s.t. $\|u\|=1$ and $\|Au\| = \sup_{\|u\|=1} \|Au\| = \|A\|$.

Applying this fact to AB, $\exists\, u \in \mathbb{R}^n$ s.t. $\|u\|=1$ and

$$\|AB\| = \|\underbrace{A}_{\text{Matrix}}\underbrace{Bu}_{\text{vec.}}\| \overset{①}{\leq} \|A\|\,\underbrace{\|Bu\|}_{\|x\|} \overset{②}{\leq} \|A\|\,\|B\|\,\underbrace{\|u\|}_{1} = \|A\|\,\|B\|.$$

By the Prop. above

$\|Mx\| \leq \|M\|\,\|x\|$

① $M=A$ $x=Bu$

② $M=B$ $x=u$

Condition Number

Consider $Ax=b$ where A is $n\times n$ and invertible.

Example

Say A^{-1} is perturbed (e.g. due to roundoff errors) to obtain a new matrix B.

Then the solution $x = A^{-1}b$ is perturbed to $\tilde{x} = Bb$.

Abs. Error $= \|x - \tilde{x}\| = \|x - Bb\| = \|x - BAx\| = \|(I-BA)x\| \leq \|I-BA\|\,\|x\|$

Thus, Relative Error $= \dfrac{\|x-\tilde{x}\|}{\|x\|} \leq \|I-BA\|$

$B \simeq A^{-1}$