Example

Given $A = \begin{pmatrix} 2 & 3 & \underline{-6} \\ 1 & -6 & \underline{8} \\ \underline{3} & -2 & 1 \end{pmatrix}$ complete the factorization phase of Gaussian elimination with scaled row pivoting. (scaled partial pivoting)

A is an $n \times n$ matrix for $n=3$.

$Ax = b$
$Ay = c$

max modulus of rows: $s_1 = \underline{6}$ $\quad s_2 = \underline{8}$ $\quad s_3 = \underline{3}$

initial perm vect: $(p_1 \; p_2 \; p_3) = (1 \; 2 \; 3)$

let t denote the step number (initially $t=1$).

Find largest $|a_{p_i t}|/s_{p_i}$ for $t \le i \le n$ (If, instead, we look for the largest $|a_{p_i t}|$, the method is called partial pivoting)

$|a_{11}|/s_1 = 2/6 \qquad |a_{21}|/s_2 = 1/8 \qquad |a_{31}|/s_3 = 3/3 = 1 \Leftarrow$ the largest

$|a_{p_i t}|/s_{p_i}$ is largest for $i=3$ so we swap p_t and p_3 in the perm vector. $p_t = 1$ $p_3 = 3$ so $(1 \; 2 \; 3)$ and we have

$(p_1 \; p_2 \; p_3) = (3 \; 2 \; 1)$

Now we can get rid of non-zero $a_{p_i t}$ for $t < i \le n$ (1; 2,3; 3)

$$\begin{array}{c} p_3 \\ p_2 \\ p_1 \end{array} \begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix}$$

Pivot element

We apply operations $R_{p_i} - \frac{a_{p_i t}}{a_{p_t t}} R_{p_t} \to R_{p_i}$ for $t < i \le n$

For $t=1$, this means ($i=2$) $R_{p_2} - \frac{a_{p_2 1}}{a_{p_1 1}} R_{p_1} \to R_{p_2}$ or $R_2 - \frac{1}{3} R_3 \to R_2$

and ($i=3$) $R_{p_3} - \frac{a_{p_3 1}}{a_{p_1 1}} R_{p_1} \to R_{p_3}$ or $R_1 - \frac{2}{3} R_3 \to R_1$

We update A:

$$A = \begin{pmatrix} 0 & 13/3 & -20/3 \\ 0 & -16/3 & 23/3 \\ 3 & -2 & 1 \end{pmatrix}$$

We can store multipliers where we have 0s: $A = \begin{pmatrix} 2/3 & 13/3 & -20/3 \\ 1/3 & -16/3 & 23/3 \\ 3 & -2 & 1 \end{pmatrix}$

Next we increase t and repeat until $t = n$.

$\boxed{t=2}$ $\boxed{\text{Find largest } |a_{p_i t}|/s_{p_i} \text{ for } t \le i \le n}$

$$|a_{p_2 2}|/s_{p_2} = |a_{22}|/s_2 = \frac{16/3}{8}$$

$$|a_{p_3 2}|/s_{p_3} = |a_{12}|/s_1 = \frac{13/3}{6} \Leftarrow$$ largest $i=3$ so we

swap p_t and p_i (p_2 and p_3) $(3 \; 2 \; 1)$. Now $(p_1 \; p_2 \; p_3) = (3 \; 1 \; 2)$

Pivot element

$$\begin{array}{c} p_2 \\ p_3 \\ p_1 \end{array} \begin{pmatrix} 2/3 & 13/3 & -20/3 \\ 1/3 & -16/3 & 23/3 \\ 3 & -2 & 1 \end{pmatrix}$$

We apply operations

$$\boxed{R_{p_i} - \frac{a_{p_i t}}{a_{p_t t}} R_{p_t} \to R_{p_i} \text{ for } t < i \le n} \quad (2 \; 3 \; 3)$$

For $t=2$ this means $R_{p_3} - \frac{a_{p_3 2}}{a_{p_2 2}} R_{p_2} \to R_{p_3}$ or $R_2 - \frac{-16/3}{13/3} R_1 \to R_2$

This is the only operation in this step since $2 = t < i \le n = 3$ and (of course) we don't apply them to the multipliers. We are storing multipliers in A only because it is convenient.

Updated A: $\begin{pmatrix} 2/3 & 13/3 & -20/3 \\ 1/3 & -16/13 & -7/13 \\ 3 & -2 & 1 \end{pmatrix}$. Increasing t by 1, we get $t = 3 = n$ so we are done!

Now

$$PA = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -16/13 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & -2 & 1 \\ 0 & 13/3 & -\frac{20}{3} \\ 0 & 0 & -7/13 \end{pmatrix}}_{U}$$

where $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\qquad A = \begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ 3 & -2 & 1 \end{pmatrix}$

$P_{ij} = \delta_{p_i j}$. $\qquad p = (3 \; 1 \; 2)$

Now that the factorization phase is over, we can carry out the solution phase as described last time.

($PAx = Pb \Rightarrow L\underbrace{Ux}_{z} = Pb$ apply forward subs to solve $Lz = Pb$ and then back subs. to solve $Ux = z$.)

Operation Counts

We will count the number of long operations or ops for short (long operations: multiplication or division.)

At step t: we find largest $|a_{p_i t}|/s_{p_i}$ for $t \le i \le n$

that gives us $n-t+1$ divisions.

We also compute $R_{p_i} - \frac{a_{p_i t}}{a_{p_t t}} R_{p_t} \to R_{p_i}$ for $t < i \le n$

Here for each i, we have 1 division due to $\frac{a_{p_i t}}{a_{p_t t}} = m$ and $n-t$ multiplication due to $m \cdot R_{p_t}$

(recall: we don't need to multiply the entire row because the first t entries correspond to multipliers which are preserved.)

Thus, we have $(n-t)^2$ multiplications as i ranges from $t+1$ to n.

$$(n-t)^2 + (n-t) + (n-t+1) = (n-t)^2 + 2(n-t) + 1 = (n-t+1)^2$$

Thus, $\sum_{t=1}^{n-1} (n-t+1)^2 = n^2 + (n-1)^2 + (n-2)^2 + \dots + 2^2 = \frac{1}{6} n(n+1)(2n+1) - 1$

$+1^2$
-1

$$= \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}-1 = O(n^3)$$

$$\approx \frac{n^3}{3}+\frac{n^2}{2}$$

Exercise: Verify that in the forward subs. step (to solve $Lz = b'$ where b' is the vector b permuted)

$$\# \text{ops} = 0+1+2+\cdots+(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n$$

In the back subs step (to solve $Ux = z$)

$$\# \text{ops} = 1+2+\cdots+n = \frac{1}{2}n^2+\frac{1}{2}n$$

(Difference in # ops is due to the fact that L is <u>unit</u> lower triang)

Thus, in the solution phase, # ops $= n^2$.

<u>Thm</u> (Theorem on Long Operations)

If Gaussian elimination is used with scaled row pivoting, then the solution of the system $Ax=b$ with fixed A and m different vectors b, involves approximately

$$\frac{1}{3}n^3 + \left(\frac{1}{2}+m\right)n^2 \quad \text{ops (mult. or division)}$$

Diagonally Dominant Matrices

<u>Def^n</u> $A_{n\times n}$ is called <u>diagonally dominant</u> if $|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad (1\le i\le n)$

$i=1$

$$A = \begin{array}{cc} i\text{th} & \\ & \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}\end{array}$$

(May arise from PDEs!)

$$1 = \frac{|a_{11}|}{|a_{11}|} > \frac{\sum_{j=2}^{n} |a_{1j}|}{|a_{11}|}$$

If A is diagonally dominant, we can use Gaussian elimination <u>without</u> pivoting.

<u>Thm</u> Gaussian elim. without pivoting preserves the diagonal dominance of the matrix.

<u>Pf</u> let $A = A^{(1)} \sim A^{(2)} \sim A^{(3)} \cdots \sim A^{(n)}$ as before.

Note that it is enough to prove $A^{(2)}$ is diagonally dominant.

We want to show $|a_{ii}^{(2)}| > \sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}^{(2)}|$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cancel{a_{21}} & & & \\ \cancel{\vdots} & & a_{ij} & \\ \cancel{a_{n1}} & \cdots & & a_{nn} \end{pmatrix}$$

$$R_i - \frac{a_{i1}}{a_{11}} R_1 \to R_i$$

Recall

$$a_{ii}^{(2)} = a_{ii} - \frac{a_{i1}}{a_{11}} a_{1i} \quad \text{and} \quad a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$$

In fact we will prove the stronger inequality

$$|a_{ii}^{(2)}| \overset{\text{tri. ineq.}}{>} |a_{ii}| - \left|\frac{a_{i1}}{a_{11}} a_{1i}\right| > \sum_{\substack{j=2\\ j\neq i}}^{n} \left(|a_{ij}| + \left|\frac{a_{i1}}{a_{11}} a_{1j}\right|\right) \overset{\text{tri. ineq.}}{>} \sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}^{(2)}|$$

or equivalently

$$|a_{ii}| - \sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}| \overset{?}{>} \sum_{j=2}^{n} \left|\frac{a_{i1}}{a_{11}} a_{1j}\right| \overset{\checkmark}{=} \frac{|a_{i1}|}{|a_{11}|} \sum_{j=2}^{n} |a_{1j}|$$

We are given

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| = |a_{i1}| + \sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}|$$

<1

$$\text{So} \quad |a_{ii}| - \sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}| > |a_{i1}| > |a_{i1}| \left(\frac{\sum_{j=2}^{n} |a_{1j}|}{|a_{11}|}\right)$$

which was to be shown.

<u>Corollary 1:</u> Every diagonally dominant matrix is nonsingular and has an LU decomposition.

<u>Pf</u> The theorem above shows that the pivot element $a_{kk}^{(k)}$ at each step is nonzero. This is enough to conclude that LU-factorization exists where L is <u>unit</u> lower triangular. In particular L is non-singular.

The theorem above also implies U is diagonally dominant. Thus, U has non-zero diagonal entries $\Rightarrow$ U is also non-singular. Therefore $A = LU$ is non-singular.

<u>Corollary 2:</u> If we applied scaled pivoting to a diagonally dominant matrix, the perm. vector would be $(1\ 2\ 3 \cdots n)$. In other words, no pivoting is necessary.

<u>Idea of Pf</u> It is clear that $s_i = |a_{ii}|$ and in the first step, $|a_{i1}|/s_i$ is maximized for $i=1$ since $|a_{11}|/s_1 = 1$ and $|a_{i1}|/s_i < 1$ for $i>1$.

(Next step is the same argument applied to the $(n-1)\times(n-1)$ submatrix at the bottom right corner.)

4.4 Norms and the Analysis of Errors

V: Vector space $\quad \|\cdot\|: V \to [0,\infty)$ is called a norm if

1) $\|x\| > 0$ for $0 \neq x \in V$
2) $\|\lambda x\| = |\lambda|\,\|x\|$ for $\lambda \in \mathbb{R},\ x\in V$
3) $\|x+y\| \le \|x\| + \|y\|$ for $x,y \in V$ (triangle inequality)