## LU-Factorizations

$\exists$ 3 (similar) variations called Doolittle's factorization, Crout's factorization and Cholesky's factorization.

LU-Factorizations are closely related to the Gaussian elimination.

Assume $A = LU$ where $L$ is lower, $U$ is upper triangular.

Then $Ax = b$ can be solved as follows:

(Note $Ax = LUx = L(Ux) = b$ )

First find $z$ s.t. $Lz = b$ (by forward subst.)

Then find $x$ s.t. $Ux = z$ (by back subst.)

When an LU-decomposition exists it is not unique!

Let $L$ be a lower triangular matrix. We say that $L$ is a <u>unit lower triangular matrix</u> if $L_{ii} = 1$ $\forall i$. Similarly if $U$ is upper triangular we say that $U$ is <u>unit upper triangular</u> if $U_{ii} = 1$ $\forall i$.

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

Say

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{array}{c} \scriptstyle i\text{th row} \\ \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & \ddots & 0000 & \vdots \\ \vdots & & \ddots & 0 \\ l_{n1} & \cdots & \cdots & l_{nn} \end{pmatrix} \end{array} \begin{array}{c} \scriptstyle j\text{th column} \\ \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix} \end{array} = LU$$

Then $l_{ij} = 0$ if $i < j$ and $u_{ij} = 0$ if $i > j$

and

$$\boxed{a_{ij}} = \sum_{k=1}^{n} l_{ik} u_{kj} = \sum_{k=1}^{\min(i,j)} l_{ik} u_{kj}$$

So $\boxed{a_{11}} = \sum_{k=1}^{1} l_{1k} u_{k1} = l_{11} u_{11}$ (By choosing $l_{11}$ or $u_{11}$ we can determine the other.)

Next, we consider $a_{12} = \sum_{k=1} l_{1k} u_{k2} = l_{11} u_{12}$ so $u_{12}$ is also determined!

Similarly, we can determine the entire first row of $U$ by

$$a_{1j} = l_{11} u_{1j}$$

Then similarly $a_{i1} = l_{i1} u_{11}$ determines the first column of $L$ uniquely.

Say we determined the first $\underbrace{r-1}_{=1}$ rows of $U$ and the first $\underbrace{r-1}_{=1}$ columns of $L$.

we have seen

Then

$$a_{rr} = \sum_{k=1}^{r} l_{rk} u_{kr}$$

$$= \underbrace{l_{r1} u_{1r} + l_{r2} u_{2r} + \cdots + l_{r\,r-1} u_{r-1\,r}}_{\text{each term is already determined by assumption}} + l_{rr} u_{rr}$$

$l_{rr} = 1$ (Doolittle's)

$u_{rr} = 1$ (Crout's)

if $l_{rr} = u_{rr}$ (Cholesky's)

choose a value for one of them to determine the other as before.

then for $t > r$

$$a_{rt} = \sum_{k=1}^{r} l_{rk} u_{kt} = \underbrace{l_{r1} u_{1t} + l_{r2} u_{2t} + \cdots + l_{r\,r-1} u_{r-1\,t}}_{\text{each term is already determined by assumption}} + \underbrace{l_{rr}}_{\text{determined above}} u_{rt}$$

So $u_{rt}$ is also determined

$\Rightarrow$ $r$th row of $U$ is completely determined.

similarly, $r$th column of $L$ can be determined.

$a_{tr}$

This completes the induction step.

Of course, the above process do not always work.

e.g. if $a_{11} = 0$ then $a_{11} = l_{11} u_{11}$ $\Rightarrow$ either $l_{11}$ or $u_{11}$ is 0.

Say $l_{11} = 0$ then if $0 \neq a_{12} = l_{11} u_{12} = 0 u_{12}$ leads to a contradiction.

If we choose $l_{ii} = 1$ $\forall i$ during the whole process, in other words if $L$ is unit lower triangular, this algorithm is known as <u>Doolittle's factorization</u>. If $u_{ii} = 1$ $\forall i$, it is known as <u>Crout's factorization</u>. When $U = L^T$ (so that $u_{ii} = l_{ii}$), the algorithm is called <u>Cholesky's factorization</u>.

Note if $U = L^T$ then $A = LU = LL^T = {L^T}^T L^T = (LL^T)^T = A^T$.

So Cholesky's factorization can only work on symmetric matrices.

In fact, $A$ needs to be <u>real</u>, <u>symmetric</u>, and <u>positive definite</u>.

<u>Example</u> Find LU factorizations of $A = \begin{pmatrix} 60 & 30 & \boxed{20} \\ \boxed{30} & 20 & 15 \\ 20 & 15 & 12 \end{pmatrix}$

Set $l_{11} = 1$ (Doolittle's factorization) $\quad a_{ij} = \sum_{k=1}^{\min(i,j)} l_{ik} u_{kj}$

then $u_{11} = a_{11} = 60$

$30 = \boxed{a_{12}} = l_{11} u_{12} = u_{12}$ $\qquad$ $20 = \boxed{a_{13}} = l_{11} u_{13} = u_{13}$

$30 = \boxed{a_{21}} = l_{21} u_{11} = 60 l_{21}$ $\qquad$ $20 = a_{31} = l_{31} u_{11} = 60 l_{31}$

hence $l_{21} = \frac{1}{2}$ $\qquad$ hence $l_{31} = \frac{1}{3}$

So far we have

$$L=\begin{pmatrix}1&0&0\\ \frac12&l_{22}&0\\ \frac13&l_{32}&l_{33}\end{pmatrix}\qquad U=\begin{pmatrix}60&30&20\\0&u_{22}&u_{23}\\0&0&u_{33}\end{pmatrix}\qquad A=LU$$

Set $l_{22}=1$, $\quad 20=\boxed{a_{22}}=30\cdot\frac12+l_{22}u_{22}=15+u_{22}$ so $u_{22}=5$

$15=a_{23}=20\cdot\frac12+l_{22}u_{23}=10+u_{23}$ so $u_{23}=5$

$15=a_{32}=30\cdot\frac13+l_{32}u_{22}=10+5\,l_{32}$ so $l_{32}=1$

finally set $l_{33}=1$ and

$12=a_{33}=20\cdot\frac13+5+l_{33}u_{33}$ so $u_{33}=\frac13$.

Hence

$$L=\begin{pmatrix}1&0&0\\ \frac12&1&0\\ \frac13&1&1\end{pmatrix}\qquad U=\begin{pmatrix}60&30&20\\0&5&5\\0&0&\frac13\end{pmatrix}$$

Note

$$A=LU=\begin{pmatrix}1&0&0\\ \frac12&1&0\\ \frac13&1&1\end{pmatrix}\begin{pmatrix}60&0&0\\0&5&0\\0&0&\frac13\end{pmatrix}\begin{pmatrix}1&\frac12&\frac13\\0&1&1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}60&0&0\\30&5&0\\20&5&\frac13\end{pmatrix}\begin{pmatrix}1&\frac12&\frac13\\0&1&1\\0&0&1\end{pmatrix}\quad\text{is Crout's factorization}$$

$$=\begin{pmatrix}1&0&0\\ \frac12&1&0\\ \frac13&1&1\end{pmatrix}\begin{pmatrix}\sqrt{60}&&\\&\sqrt5&\\&&\sqrt{\frac13}\end{pmatrix}^2\begin{pmatrix}1&\frac12&\frac13\\0&1&1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt{60}&0&0\\ \frac{\sqrt{60}}{2}&\sqrt5&0\\ \frac{\sqrt{60}}{3}&\sqrt5&\sqrt{\frac13}\end{pmatrix}\begin{pmatrix}\sqrt{60}&\frac{\sqrt{60}}{2}&\frac{\sqrt{60}}{3}\\0&\sqrt5&\sqrt5\\0&0&\sqrt{\frac13}\end{pmatrix}=LL^T$$

From now on our "default" LU-factorization is Doolittle's factorization i.e. $L$ is unit lower triangular.

If $A=\begin{pmatrix}a_{11}&\cdots&a_{1k}&\cdots&a_{1n}\\ \vdots&&\vdots&&\vdots\\ a_{k1}&\cdots&a_{kk}&&\vdots\\ \vdots&&&&\\ a_{n1}&\cdots&\cdots&\cdots&a_{nn}\end{pmatrix}$ then the $k^{th}$ leading principal minor is

$$A_k=\begin{pmatrix}a_{11}&\cdots&a_{1k}\\ \vdots&&\vdots\\ a_{k1}&\cdots&a_{kk}\end{pmatrix}$$

**Thm** If all $n$ leading principal minors of the $n\times n$ matrix $A$ are nonsingular, then $A$ has an LU-decomposition.

**Idea of the proof**

$a_{11}=l_{11}u_{11}$ LPM: leading principal minor

Note if

$$\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&&\vdots\\ \vdots&&&\vdots\\a_{n1}&\cdots&\cdots&a_{nn}\end{pmatrix}=\begin{pmatrix}l_{11}&0&\cdots&0\\l_{21}&l_{22}&0\cdots&0\\ \vdots&&\ddots&0\\l_{n1}&\cdots&\cdots&l_{nn}\end{pmatrix}\begin{pmatrix}u_{11}&u_{12}&\cdots&u_{1n}\\0&u_{22}&&\vdots\\0&0&\ddots&\vdots\\0&0\cdots&0&u_{nn}\end{pmatrix}$$

that is $A_1=L_1U_1$, $A_2=L_2U_2$ and so on.

By assumption, $A_1$ is invertible and we chose $L_1=(1)$ and thus, $U_1=A_1$.

Now assume we determined $L_{r-1}$ and $U_{r-1}$. Since $A_{r-1}=L_{r-1}U_{r-1}$ is invertible, $L_{r-1}$ and $U_{r-1}$ are invertible.

$$\begin{pmatrix}a_{11}&&a_{ir}&&a_{1n}\\&a_{r-1,r-1}&&&\\&&a_{rr}&&\\&&\vdots&&\\a_{n1}&&&&a_{nn}\end{pmatrix}=\begin{array}{c}i\text{th row}\\ \begin{pmatrix}l_{11}&0&\cdots&&0\\ &&0&1&0\\&&l_{r-1,r-1}&&\\&&l_{rr}&0\cdots&0\\&&&&0\\l_{n1}&&&&l_{nn}\end{pmatrix}\end{array}\begin{array}{c}r\text{th col}\\ \begin{pmatrix}u_{11}&&&&u_{1n}\\0&&&&\\&0\;u_{r-1,r-1}&&&\\&&u_{rr}&&\\0&&0&&\\&&0&\cdots 0&u_{nn}\end{pmatrix}\end{array}$$

$a_{ir}=$ ( $i$th row of $L$ ) $\cdot$ ( $j$th Column of $U$ ).

Recall if $B$ is invertible $Bx=b$ has a uniq. solution $x=B^{-1}b$

$$\begin{pmatrix}l_{11}&0&\cdots&0\\ \vdots&&&\\ &&&0\\ l_{r-1,1}&&&l_{r-1,r-1}\end{pmatrix}\begin{pmatrix}u_{1r}\\u_{2r}\\ \vdots\\u_{r-1,r}\end{pmatrix}=\begin{pmatrix}a_{1r}\\a_{2r}\\ \vdots\\a_{ir}\\ \vdots\\a_{r-1,r}\end{pmatrix}$$

$L_{r-1}$ is determined and invertible $\Rightarrow\begin{pmatrix}u_{1r}\\ \vdots\\u_{r-1,r}\end{pmatrix}$ is uniquely determined

Similarly, $(l_{r1}\;l_{r2}\;\cdots\;l_{r,r-1})$ is unique.

Next we choose $l_{rr}=1$ and since

$a_{rr}=l_{r1}u_{1r}+l_{r2}u_{2r}+\cdots+l_{r,r-1}u_{r-1,r}+l_{rr}u_{rr}$, $u_{rr}$ is uniquely determined! This completes the induction step.

**Thm** Cholesky Theorem on $LL^T$ factorizations

If $A$ is a real, symmetric and pos. def. matrix, then it has a unique factorization, $A=LL^T$ in which $L$ is lower triangular with positive diagonal entries.