Last time $x_{n+1} = F(x_n)$ assume $\lim_{n\to\infty} x_n = p$

and $F'(p) = F''(p) = \cdots = F^{(q-1)}(p) = 0$ but $F^{(q)}(p) \neq 0$

then $|e_{n+1}| \le C|e_n|^q$ $\qquad \dfrac{F^{(q)}(p)}{q!} < C$

In other words, the order of convergence is q.

e.g. If $F'(p) = 0$ and $F''(p) \neq 0$ then

$$e_{n+1} = \frac{1}{2} F''(y_n)\, e_n^2 \quad (\text{similar to Newton's Method})$$

In Newton's method, $F(x) = x - \dfrac{f(x)}{f'(x)}$.

So
$$F'(x) = 1 - \frac{f'f' - ff''}{(f')^2} = \frac{(f')^2 - (f')^2 + ff''}{(f')^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

Since $\underset{\text{(root of } f)}{f(p) = 0}$, $F'(p) = 0$.

In general, $F''(x) = \dfrac{(f'f'' + ff''')(f')^2 - 2f'f''\,ff''}{(f')^4}$

$f(p) = 0$

and $F''(p) = \dfrac{(f'(p))^3 f''(p)}{(f'(p))^4} = \dfrac{f''(p)}{f'(p)}$ need not be 0.

Chp4 Solving Systems of Linear Eq.s

4.1 Matrix Algebra

Consider
$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{matrix}$$
n equations in n unknowns

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

then the system is $Ax = b$

Recall
If $A_{m\times p}$ and $B_{p\times n}$ then $(AB)_{m\times n}$ and
$$(AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \quad \text{for } 1 \le i \le m,\ 1 \le j \le n$$

Prop The set of all $n\times n$ $\mathbb{C}$ matrices is a $\mathbb{C}$ algebra.

Notation Elementary Row Operations

1) $A_i \leftrightarrow A_j$ interchanging ith and jth rows $\quad R_i \leftrightarrow R_j$
2) $\lambda A_i \to A_i$ multiplying ith row by $\lambda \neq 0$
3) $A_i + \lambda A_j \to A_i$ adding to the ith row λ times the jth row.

We write $A \sim B$ if B is obtained from A by a sequence of E.R.O.s, and say "A is row equivalent to B".

E is called an elementary matrix if E is obtained from I by applying a single E.R.O. $\qquad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

Note that applying a (single) E.R.O to a matrix A to obtain B is equivalent to multiplying A from left by the corr. elem. matrix E, that is $B = EA$.

e.g. Say $A = \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} \overset{2A_2 + A_1 \to A_1}{\sim} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = B$. $\quad I \overset{2A_2 + A_1 \to A_1}{\sim} E = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

so $EA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = B$

Recall If $A \sim B$ then $\det(A) = 0 \iff \det B = 0$

Moreover if $\det A \neq 0$, then $A \sim I$.

So $\exists$ elem. matrices $E_1, E_2, \ldots, E_k$ s.t.
$$(E_k E_{k-1} \cdots E_2 E_1) A = I$$
Thus, $A^{-1} = (E_k E_{k-1} \cdots E_1)$ $\qquad \left(A = (A^{-1})^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \right)$

Thm For an $n\times n$ matrix A, the following properties are equivalent:

1) A^{-1} exists (i.e. A is non-singular).
2) $\det A \neq 0$.
3) rows of A form a basis for $\mathbb{R}^n$.
4) columns of A form a basis for $\mathbb{R}^n$.
5) As a map $(\mathbb{R}^n \to \mathbb{R}^n)$, A is 1-1. $\qquad x \in \mathbb{R}^n$, $x \longmapsto Ax \in \mathbb{R}^n$
6) As a map $(\mathbb{R}^n \to \mathbb{R}^n)$, A is onto.
7) $Ax = 0 \Rightarrow x = 0$ $\qquad (\text{nullspace}(A) = \{0\})$
8) for all $b \in \mathbb{R}^n$, $\exists!$ (unique) x s.t. $Ax = b$
9) A is a prod. of elem. matrices.
10) 0 is not an eigenvalue of A. $\qquad Ax \overset{\neq 0}{=} 0 \cdot x = 0$

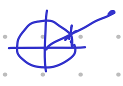
> $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $(x\ y) A \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$
> In general if A is positive def., we can declare the norm of $x \in \mathbb{R}^n$ to be
> $|x| = \sqrt{x^T A x}$

Def^n $A_{n\times n}$ is called positive definite if $x^T A x > 0$ $\forall x \in \mathbb{R}^n$, $x \neq 0$ (for all)

positive semi-definite if $x^T A x \ge 0$ $\forall x \in \mathbb{R}^n$

e.g.
$$(x\ y)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x\ y)\begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix} = 2x^2 + xy + xy + 2y^2 = x^2 + (x+y)^2 + y^2 > 0$$
if $(x, y) \neq (0, 0)$.

$z = re^{i\theta}$
$r \geq 0$



Partitioned Matrices

Say $A = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ \hline 0 & 2 & -1 \\ -1 & 5 & 0 \end{array}\right)_{3\times 3}$ $\quad B = \left(\begin{array}{c|c} 2 & -3 \\ 4 & 0 \\ \hline 5 & 1 \end{array}\right)_{3\times 2}$

so $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ where

$A_{11} = (1\ 2)$ $\quad A_{12} = (1)$ $\quad B_{11} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\quad B_{12} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$A_{21} = \begin{pmatrix} 0 & 2 \\ -1 & 5 \end{pmatrix}$ $\quad A_{22} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\quad B_{21} = (5)$ $\quad B_{22} = (1)$

Then $AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$

$$= \begin{pmatrix} (1\ 2)\begin{pmatrix} 2 \\ 4 \end{pmatrix} + (1)(5) & (1\ 2)\begin{pmatrix} -3 \\ 0 \end{pmatrix} + (1)(1) \\ \begin{pmatrix} 0 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}(5) & \begin{pmatrix} 0 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}(1) \end{pmatrix}$$

$$= \begin{pmatrix} 2+8+5 & -3+0+1 \\ 0+8-5 & 0+0-1 \\ -2+20+0 & 3+0+0 \end{pmatrix} = \begin{pmatrix} 15 & -2 \\ 3 & -1 \\ 18 & 3 \end{pmatrix} = AB$$

In general we have,

Thm If $A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$ $\quad B = \begin{pmatrix} B_{11} & \cdots & B_{1k} \\ \vdots & & \\ B_{n1} & \cdots & B_{nk} \end{pmatrix}$

$C = \begin{pmatrix} C_{11} & \cdots & C_{1k} \\ \vdots & & \vdots \\ C_{m1} & \cdots & C_{mk} \end{pmatrix}$

where A_{**}, B_{**}, and C_{**} are submatrices so that each product $A_{is}B_{sj}$ can be formed, and

$C_{ij} = \sum_{s=1}^{n} A_{is}B_{sj}$ then $C = AB$.

4.2 LU and Cholesky Factorizations

Say we have a diagonal matrix A. Then it is easy to solve $Ax = b$ (if there is a solution).

$$\begin{aligned} a_{11}x_1 \quad &= b_1 \\ a_{22}x_2 \quad &= b_2 \\ \ddots \quad a_{nn}x_n &= b_n \end{aligned}$$

so $x_i = \frac{b_i}{a_{ii}}$ (if $a_{ii} \neq 0$)

if $a_{ii} = 0$ and $b_i = 0$ x_i can be anything (∞'ly many solutions)

// and $b_i \neq 0$ no solution exists.

If A is lower triangular, $\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & & 0 \\ \vdots & & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{pmatrix}$

$$\begin{aligned} a_{11}x_1 \quad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \quad &= b_2 \\ \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

We can solve by forward substitution:

First, $x_1 = b_1/a_{11}$

then plug it in the second eq.

and we have $\quad x_2 = \left(b_2 - a_{21}x_1\right)/a_{22}$

and then $\quad x_3 = \left(b_3 - a_{31}x_1 - a_{32}x_2\right)/a_{33}$ and so on.

In general we determine x_i after x_{i-1} and

$$x_i = \left(b_i - \left(\sum_{j=1}^{i-1} a_{ij}x_j\right)\right)/a_{ii}$$

Finally if A is upper triangular, $\begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots 0 & * \end{pmatrix}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned}$$

We use back substitution

$x_n = b_n/a_{nn}$,

$x_{n-1} = \left(b_{n-1} - a_{n-1\,n}x_n\right)/a_{n-1\,n-1}$

and so on

In general x_i is determined before x_{i-1} and

$$x_i = \left(b_i - \left(\sum_{j=i+1}^{n} a_{ij}x_j\right)\right)/a_{ii}$$

Say we have $\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \rightarrow (a_{31}x_1 = b_3)$

Note that the system is equivalent to

$$\begin{pmatrix} a_{31} & 0 & 0 \\ a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix}$$

We can solve this system by forward substitution.

Note that "we applied the permutation $(3\ 1\ 2)$".

In general if we have a "permuted lower triangular system" with permutation vector $(p_1\ p_2\ p_3\ \cdots\ p_n)$ the algorithm to solve the system is

for $i = 1$ to n do

$$x_i \leftarrow \left(b_{p_i} - \sum_{j=1}^{i-1} a_{p_i j}\, x_j\right) \Big/ a_{p_i i}$$

end do

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$(p_1\ p_2\ p_3) = (3\ 1\ 2)$

$x_1 = b_{p_1} / a_{p_1 1} = b_3 / a_{31}$

Of course, we change back substitution method in similar ways to solve "permuted upper triangular system".

LU-Factorizations

$\exists$ 3 (similar) variations called Doolittle's factorization, Crout's factorization and Cholesky's factorization.

LU-Factorizations are closely related to the Gaussian elimination.

Assume $A = LU$ where L is lower, U is upper triangular.

Then $Ax = b$ can be solved as follows:

(Note $Ax = LUx = L(Ux) = b$)

First find z s.t. $Lz = b$ (by forward subst.)

Then find x s.t. $Ux = z$ (by back subst.)

When an LU-decomposition exists it is not unique.