Order of Convergence of the Secant Method

Recall: $e_{n+1} \approx \frac{1}{2}\frac{f''(r)}{f'(r)} e_n e_{n-1} = C e_n e_{n-1}$ $\quad C = \frac{1}{2}\frac{f''(r)}{f'(r)}$

Assume $|e_{n+1}| \sim A|e_n|^\alpha$, $A>0$ i.e. $\lim_{n\to\infty} \frac{|e_{n+1}|}{A|e_n|^\alpha} = 1$

In other words we are assuming that the order of convergence is α.

Note $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = A \lim_{n\to\infty} \frac{|e_{n+1}|}{A|e_n|^\alpha} = A$

Thus,
$$\lim_{n\to\infty} \frac{\left(\frac{1}{A}|e_{n+1}|\right)^{1/\alpha}}{|e_n|} = \frac{1}{A^{1/\alpha}} \lim_{n\to\infty} \left(\frac{|e_{n+1}|}{|e_n|^\alpha}\right)^{1/\alpha} = \frac{1}{A^{1/\alpha}}\left(\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha}\right)^{1/\alpha}$$
$$= \frac{1}{A^{1/\alpha}} A^{1/\alpha} = 1$$

In other words, $|e_n| \sim \left(\frac{1}{A}|e_{n+1}|\right)^{1/\alpha}$

or $|e_{n-1}| \sim \left(\frac{1}{A}|e_n|\right)^{1/\alpha}$

Thus $|e_{n+1}| \sim |C||e_n||e_{n-1}|$ can be rewritten as

$$A|e_n|^\alpha \sim |C||e_n| A^{-1/\alpha}|e_n|^{1/\alpha} = \frac{|C|}{A^{1/\alpha}}|e_n|^{1+1/\alpha}$$

$$\frac{A^{1+1/\alpha}}{|C|} \sim |e_n|^{1-\alpha+\frac{1}{\alpha}}$$

↑ Const $\quad$ but $e_n \to 0$ so we must have $1-\alpha+\frac{1}{\alpha}=0$

or $\alpha^2-\alpha-1=0 \rightarrow \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62$ is the positive root

So $|e_{n+1}| \sim A|e_n|^{1.62}$ better than linear but worse than quadratic convergence.

Note that $\frac{A^{1+1/\alpha}}{|C|} \sim |e_n|^{1-\alpha+\frac{1}{\alpha}=0} = 1$

Thus, $A = |C|^{\frac{1}{1+1/\alpha}}$ but also $1-\alpha+\frac{1}{\alpha}=0 \Rightarrow \alpha = 1+\frac{1}{\alpha}$

So $A = |C|^{1/\alpha} \approx |C|^{0.62} = \left|\frac{1}{2}\frac{f''(r)}{f'(r)}\right|^{0.62}$

Recall: a single step of the secant method $\quad$ 1 fn evaluation

a single step of the Newton's method $\quad$ 2 fn evaluations

In two steps of the secant method $\quad |e_{n+2}| \sim A|e_{n+1}|^\alpha \sim A^2|e_n|^{\alpha^2} = A^2|e_n|^{2.62}$

This is better than quadratic!

3.4 Fixed Points and Functional Iteration

Consider a procedure of the form $x_{n+1} = F(x_n)$ $(n \geq 0)$

e.g. Newton's method on a fixed function f.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = F(x_n)$$

Such an algorithm is called functional iteration.

We will analyze algorithms of this form (which converge).

Assume $\lim_{n\to\infty} x_n = p$ and F is continuous.

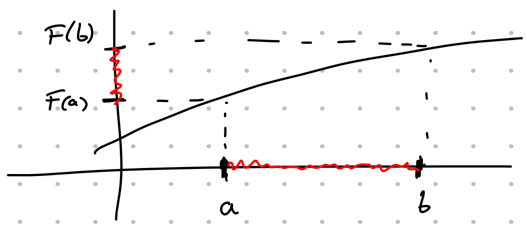
$F(p) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = p$

$F(p) = p$ i.e. p is a fixed point of F.

Def^n $F: [a,b] \to \mathbb{R}$ is said to be contractive if
$$|F(x)-F(y)| \leq \lambda|x-y|$$
for some $\lambda < 1$ for all $x,y \in [a,b]$

$\forall \varepsilon > 0$ choose $\delta = \varepsilon > 0$ then F is cont.

"contractive" $\Rightarrow$ "continuous"

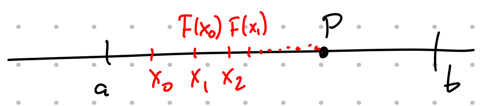


Thm Contractive Mapping Theorem

If $F: [a,b] \to [a,b]$ is contractive, then it has a unique fixed point p.

Moreover if $x_0 \in [a,b]$ and $x_{n+1} = F(x_n)$ $(n \geq 0)$

then $\lim_{n\to\infty} x_n = p$



Pf: First note that
$$|x_n - x_{n-1}| = |F(x_{n-1}) - F(x_{n-2})| \leq \lambda|x_{n-1}-x_{n-2}|$$

Thus, $|x_n - x_{n-1}| \leq \lambda|x_{n-1}-x_{n-2}| \leq \lambda^2|x_{n-2}-x_{n-3}| \leq \cdots \leq \lambda^{n-1}|x_1-x_0|$

Now $x_n = x_0 + (x_1-x_0) + (x_2-x_1) + \cdots + (x_n - x_{n-1})$

So x_n conv. iff $\sum_{k=1}^{\infty} x_k - x_{k-1}$ conv.

Recall that if $\sum_{n=1}^{\infty}|a_n|$ conv. then $\sum_{n=1}^{\infty} a_n$ conv.

$$\sum_{k=1}^{\infty} |x_k - x_{k-1}| \le \sum_{k=1}^{\infty} \lambda^{k-1}|x_1 - x_0| = \frac{1}{1-\lambda}|x_1 - x_0|$$

That is $\sum_{k=1}^{n} |x_k - x_{k-1}|$ is a monotonic and bounded seq. Therefore it converges and x_n also conv.. Say $x_n \to p$

We have already seen that $F(p) = p$. (Is this valid?)

Yes contractive mappings are always continuous.

Say there are two fixed points p_1 and p_2. Then

$$|p_1 - p_2| = |F(p_1) - F(p_2)| \le \lambda |p_1 - p_2|$$

So $(1-\lambda)|p_1 - p_2| \le 0$ $\quad 1-\lambda > 0 \Rightarrow |p_1 - p_2| = 0$

or $p_1 = p_2$. □

Claim: $\{x_n\}$ is a Cauchy seq. i.e. $\forall \varepsilon > 0 \ \exists N$ s.t. $|x_n - x_m| < \varepsilon$ for $n, m \ge N$.

$\sqrt{2} = 1.414\ldots$

$x_0 = 1 \quad x_1 = 1.4 \quad x_2 = 1.41 \ldots$

$x_n \in \mathbb{Q} \quad x_n \to \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Pf (Say $n \ge m \ge N$

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m|$$

$$\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$

$$\le \lambda^{n-1}|x_1 - x_0| + \lambda^{n-2}|x_1 - x_0| + \cdots + \lambda^m |x_1 - x_0|$$

$$= \lambda^m |x_1 - x_0| (\lambda^{n-1-m} + \cdots + \lambda^2 + \lambda + 1)$$

$$\le \lambda^m |x_1 - x_0| \left(\sum_{k=0}^{\infty} \lambda^k\right) = \frac{\lambda^m}{1-\lambda}|x_1 - x_0| \le \frac{\lambda^N}{1-\lambda}|x_1 - x_0|$$

Given $\varepsilon > 0$ $\exists N$ s.t. $\frac{\lambda^N}{1-\lambda}|x_1 - x_0| < \varepsilon$. Thus,

$|x_m - x_n| < \varepsilon$ for $n, m \ge N$.

Example Prove that $\{x_n\}$ defined by

$x_0 = -15 \qquad x_{n+1} = 3 - \frac{1}{2}|x_n| \quad (n \ge 0)$ is conv.

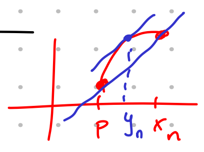
Pf $F(x) = 3 - \frac{1}{2}|x|$.

$$|F(x) - F(y)| = |\cancel{3} - \tfrac{1}{2}|x| - (\cancel{3} - \tfrac{1}{2}|y|)| = \tfrac{1}{2}\big||y| - |x|\big|$$

$\le \frac{1}{2}|y - x|$ (why?) (triangle inequality)

Q) What is the fixed point for the above example?

A) Set $F(x) = x$ $\quad F(x) = 3 - \frac{1}{2}|x| = x$ $\quad$ So if $x \ge 0$ then $3 - \frac{1}{2}x = x$ or $3 = \frac{3}{2}x$ and $x = 2$ is the unique fxd point.



Error Analysis

Set $e_n = x_n - p$. Suppose F is infinitely differentiable.

$$e_{n+1} = x_{n+1} - p = F(x_n) - F(p) = F'(y_n)(x_n - p) = F'(y_n)e_n$$

by the MVT (for some y_n between x_n and p).

Since $|F'(x)| < 1$ (why?), $\quad \left|\frac{F(x) - F(y)}{x - y}\right| \le \lambda < 1$

the error terms get smaller in magnitude. $e_{n+1} = F'(y_n)e_n$

In general convergence is linear ($|e_{n+1}| \le \lambda |e_n|$)

Clearly $F'(y_n) \to F'(p)$ since $y_n \to p$.

Thus the convergence might be faster if, for example, $F'(p) = 0$.

More generally, say $F'(p) = F''(p) = \cdots = F^{(q-1)}(p) = 0$ and $F^{(q)}(p) \ne 0$.

Then

$$e_{n+1} = x_{n+1} - p = F(x_n) - F(p) = F(p + e_n) - F(p)$$

$$= \left(\cancel{F(p)} + e_n F'(p) + \tfrac{1}{2}e_n^2 F''(p) + \cdots + \frac{e_n^{q-1}}{(q-1)!}F^{(q-1)}(p) + \frac{e_n^q}{q!}F^{(q)}(y_n)\right) - \cancel{F(p)}$$

remainder term in Taylor's thm.

$$e_{n+1} = \frac{F^{q}(y_n)}{q!}e_n^q$$ for some y_n between p and x_n.

Since $\frac{F^q(y_n)}{q!} \to \frac{F^q(p)}{q!}$ $\quad \exists C$ s.t. $|e_{n+1}| \le C|e_n|^q$

(if it is conv. at all.) In other words, the order of convergence is q.