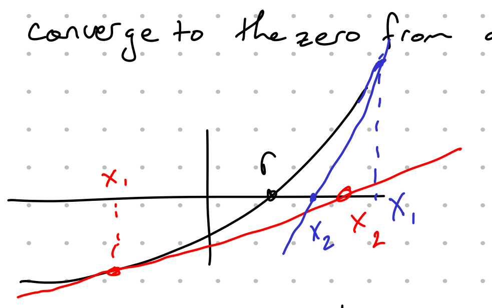
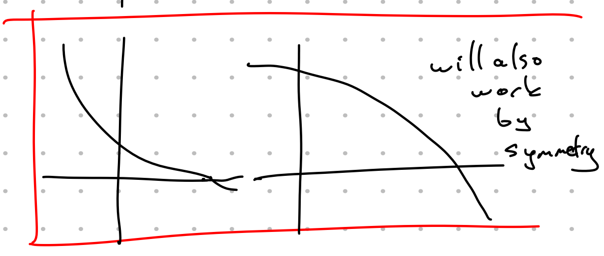
Thm Newton's Method for a Convex Function $(f''(x) > 0)$

If $f \in C^2(\mathbb{R})$, is increasing, is convex, and has a zero then the zero is unique and the Newton iteration will converge to the zero from any initial point.





Pf Note that $f'(x) > 0$, $f''(x) > 0$ $\forall x \in \mathbb{R}$.

Recall $e_{n+1} = \frac{1}{2}\frac{f''(y_n)}{f'(x_n)} e_n^2$. Thus $e_{n+1} = x_{n+1} - r > 0$ $\quad n \geq 1$

Thus, $\quad x_n > r \quad (n \geq 2)$

and $\quad f(x_n) > f(r) = 0 \quad$ (since f is inc.)

Recall

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} \quad \text{So} \quad e_{n+1} < e_n$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad e_n = x_n - r}$$

Thus e_n (for $n \geq 1$) is a strictly decreasing sequence of positive numbers. Thus, e_n converges.

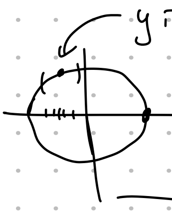
Similarly x_n (for $n \geq 1$) is also strictly decr. seq. with $x_n > r$. Thus, x_n also converges. $\left(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\right)$

$$\cancel{\lim_{n\to\infty} e_{n+1}} = \cancel{\lim_{n\to\infty} e_n} - \frac{f(\lim_{n\to\infty} x_n)}{f'(\lim_{n\to\infty} x_n)} = 0$$

$$\Rightarrow f\left(\lim_{n\to\infty} x_n\right) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} x_n = r$$

Implicit functions

Consider $\quad x^2 + y^2 - 1 = 0$



y is a function of x.

$y = y(x) = \sqrt{1-x^2}$

so $\quad x^2 + (y(x))^2 - 1$

$\quad = x^2 + 1 - x^2 - 1 = 0$

In general $G(x,y) = 0$ may define $y = y(x)$ s.t. $G(x, y(x)) = 0$

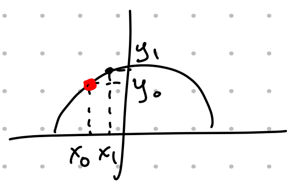
For a fixed x, we can find $y(x)$ using Newton's method.

$$y_{n+1} = y_n - \frac{G(x,y_n)}{G_y(x,y_n)} \quad \text{where} \quad G_y = \frac{\partial G}{\partial y}$$

Say $\quad y(x_0) = y_0 \quad$ i.e. $G(x_0, y_0) = 0$

and we want to compute nearby (x_1, y_1) s.t. $G(x_1, y_1) = 0$

We can use y_0 as the first approx. to y_1.



for "reasonable" G, the iteration for $x = x_1$ starting with $y = y_0$ will rapidly yield y_1. By continueing this way, we can graph $G = 0$ quickly.

See Example 3 from the textbook.

Systems of Nonlinear Equations

Say we want to solve the system

$$\begin{cases} f_1(x_1,x_2) = 0 \\ f_2(x_1,x_2) = 0 \end{cases}$$

Assume (x_1, x_2) is an approximate solution then for some h_1, h_2

$$0 = f_1(x_1+h_1, x_2+h_2) \approx f_1(x_1,x_2) + h_1 \frac{\partial f_1}{\partial x_1} + h_2 \frac{\partial f_1}{\partial x_2}$$

$$0 = f_2(x_1+h_1, x_2+h_2) \approx f_2(x_1+x_2) + h_1 \frac{\partial f_2}{\partial x_1} + h_2 \frac{\partial f_2}{\partial x_2}$$

By Taylor's thm

let $J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$ Then we have

In 1d case this means $f'(x_n) \neq 0$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} + J\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0$$

So $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -J^{-1}\begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$ if J is invertible

let X_n denote our n^{th} approx. to the solution of

$F(X) = 0$ where $F = (f_1, f_2)$ $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then if we set $J(x_1,x_2) =: F'(X)$,

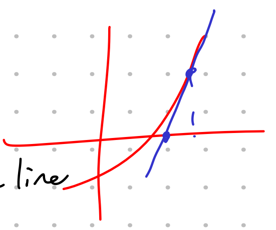
$$X_{n+1} = X_n + H = X_n - J^{-1}(X_n)F(X_n)$$

$$\text{or} \quad X_{n+1} = X_n - (F'(X_n))^{-1}F(X_n)$$

See Example 4.

In general we can use a very similar iteration for n (non-linear) equations in n unknowns.

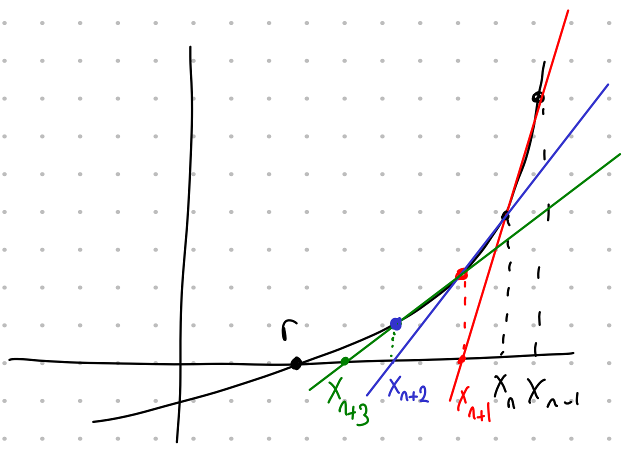
3.3 Secant Method



Newton's method: Tangent line at $x_n \approx f$. Use the line to approximate 0.

- Requires existence (and computation) of f'.

Secant's Method: Use secant line through $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ to find $(x_{n+1}, f(x_{n+1}) \approx 0)$



Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

Secant method:
$$x_{n+1} = x_n - \frac{f(x_n)\,(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Note this does not mean we compute f more than once in each step! We can compute $f(x_{n-1})$ and store it during the $n-1^{st}$ step to use it in the n^{th} step!

On the other hand, Newton's method requires computation of f and f' in each step.

So typically we do twice as much work in a single step of Newton's method as a single step of Secant Method.

Example Find the zero of $f(x) = -x^3 + x + 5$ starting with $x_0 = 1$ and $x_1 = 2$.

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$f(x_0) = 5$

$f(x_1) = -1$

$f(x_2) \approx 0.67130$ $\quad x_2 = 2 - \dfrac{(-1)(2-1)}{-1-5} = 2 + \dfrac{1}{-6} = \dfrac{11}{6}$

$f(x_3) \approx 3.8277 \cdot 10^{-2}$

$f(x_4) \approx 1.6201 \cdot 10^{-3}$ $\quad x_3 = \dfrac{11}{6} - \dfrac{0.67130\left(\frac{11}{6} - 2\right)}{0.67130 - (-1)} \approx 1.9003$

$f(x_5) \approx 3.6442 \cdot 10^{-6}$ $\quad \vdots$

$f(x_6) \approx 3.4563 \cdot 10^{-10}$

$f(x_7) \approx 0$ $\quad x_7 \approx 1.9042$

Error Analysis

Let $e_n = x_n - r$. Then

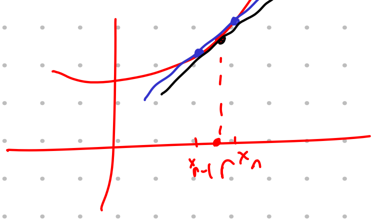
$$e_{n+1} = x_{n+1} - r = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} - r$$

$$= \frac{\cancel{f(x_n)x_n} - f(x_{n-1})x_n - \cancel{f(x_n)x_n} + \boxed{f(x_n)x_{n-1} - f(x_n)r} + f(x_{n-1})r}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)(x_{n-1} - r) - f(x_{n-1})(x_n - r)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})} = \frac{\left[\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}\right]}{f(x_n) - f(x_{n-1})} e_n e_{n-1}$$

$$= \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} e_n e_{n-1}$$



$$\approx \frac{1}{f'(r)} \quad \frac{1}{2}f''(r) \quad e_n e_{n-1} = \frac{1}{2}\frac{f''(r)}{f'(r)} e_n e_{n-1} \approx e_{n+1}$$

Why does $\dfrac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \to \dfrac{1}{2}f''(r)$?

$e_{n+1} \approx C e_n e_{n-1}$

$$f(x_n) = f(r + e_n) = f(r) + e_n f'(r) + \frac{1}{2}e_n^2 f''(r) + O(e_n^3)$$

Since $f(r) = 0$, $\quad f(x_n) \approx e_n f'(r) + \frac{1}{2}e_n^2 f''(r)$

$$\frac{f(x_n)}{e_n} \approx f'(r) + \frac{1}{2}e_n f''(r)$$

and

$$\frac{f(x_{n-1})}{e_{n-1}} \approx f'(r) + \frac{1}{2}e_{n-1} f''(r)$$

So
$$\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}} \approx \cancel{f'(r)} + \frac{1}{2}e_n f''(r) - \left[\cancel{f'(r)} + \frac{1}{2}e_{n-1} f''(r)\right]$$

$$f(x_n)/e_n - f(x_{n-1})/e_{n-1} \approx \frac{1}{2}f''(r)(e_n - e_{n-1})$$

$$\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \approx \frac{1}{2}f''(r)$$

$e_n = x_n - r$ so

$e_n - e_{n-1} = x_n - r - (x_{n-1} - r) = x_n - x_{n-1}$

$$\boxed{e_{n+1} \approx \frac{1}{2}\frac{f''(r)}{f'(r)} e_n e_{n-1} = C e_n e_{n-1}}$$

To find order of convergence: we assume $|e_{n+1}| \sim A|e_n|^\alpha$