3.2 Newton's Method

In general faster than the bisection method but does not always converge.

Setup: Suppose f'' exists and is continuous.

Let r be a zero of f and x be an approximation to r. Set $h = r - x$. Then,

$$0 = f(r) = f(x+h) = f(x) + hf'(x) + O(h^2)$$

if h is small enough, $O(h^2) \approx 0$ and

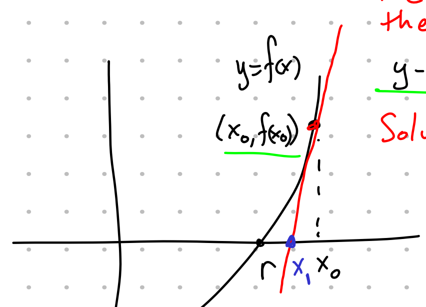
$$f(x) + hf'(x) \approx 0 \quad \text{so } h \approx -\frac{f(x)}{f'(x)}$$

$$\text{so } r = x + h \approx x - \frac{f(x)}{f'(x)}$$

The idea is: If x is an approx. to r, $x - \frac{f(x)}{f'(x)}$ is a better approx. to r.

We start with a guess x_0 and inductively define $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n \geq 0$

Geometrically:



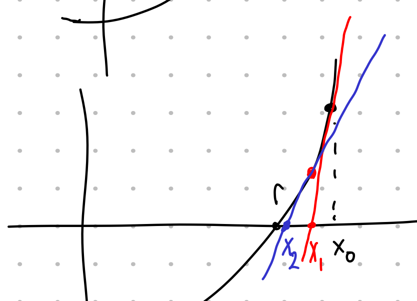
the equation of the line is:

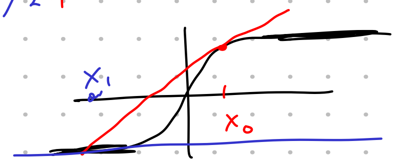
$$y - f(x_0) = f'(x_0)(x - x_0)$$

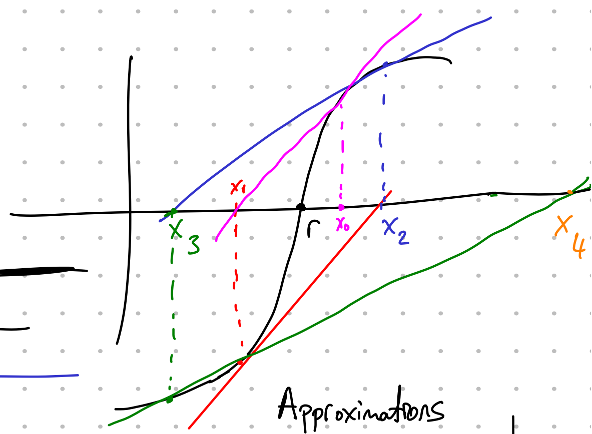
Solving for $y=0$, we get x_1

$$-f(x_0) = f'(x_0)(x_1 - x_0) = f'(x_0)x_1 - f'(x_0)x_0$$

$$f'(x_0)x_1 = f'(x_0)x_0 - f(x_0) \qquad \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$







Approximations are getting worse!

Pseudocode

```
input f, f', x0, M, δ, ε
v ← f(x0)
output 0, x0, v
if |v| < ε then stop
for k = 1 to M do
    x1 ← x0 - v/f'(x0)
    v ← f(x1)
    output k, x1, v
    if |x1 - x0| < δ or |v| < ε then stop
    x0 ← x1
end do
```

$$\boxed{\begin{aligned} x_0 &= c \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = G(x_n) \end{aligned}}$$

Example Find the zero of $f(x) = -x^3 + x + 5$ starting with $x_0 = 1$.

$$f'(x) = -3x^2 + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{-x_n^3 + x_n + 5}{3x_n^2 - 1}$$

$$x_1 = 1 + \frac{-1+1+5}{2} = \frac{7}{2} = 3.5$$

$$x_2 = 3.5 + \frac{-(3.5)^3 + 3.5 + 5}{3(3.5)^2 - 1} \approx 2.53846$$

$\vdots$

$f(x_0) = 5$

$f(x_1) = -34.375$

$f(x_2) \approx -8.8188$

$f(x_3) \approx -1.6512$

$f(x_4) \approx -1.2014 \cdot 10^{-1}$ $\quad f(x_5) \approx -8.2545 \cdot 10^{-3}$

$f(x_6) \approx -3.9888 \cdot 10^{-8}$ $\quad f(x_7) \approx 0$

$x_7 \approx 1.9042$

Error Analysis

Let $e_n = x_n - r$ denote the errors. Suppose $f(r) = 0 \neq f'(r)$ and f'' is cont. Then,

$$e_{n+1} = x_{n+1} - r = x_n - \frac{f(x_n)}{f'(x_n)} - r = x_n - r - \frac{f(x_n)}{f'(x_n)}$$

$$= e_n - \frac{f(x_n)}{f'(x_n)} = \frac{\boxed{e_n f'(x_n) - f(x_n)}}{f'(x_n)}$$

$$f(x+h) = f(x) + hf'(x) + E_1(h)$$

On the other hand

$$0 = f(r) = f(x_n - e_n) = \boxed{f(x_n) - e_n f'(x_n)} + \boxed{\frac{1}{2}e_n^2 f''(y_n)}$$

for some y_n between r and x_n by Taylor's Thm.

Thus $\boxed{e_n f'(x_n) - f(x_n)} = \frac{1}{2}e_n^2 f''(y_n)$

and $e_{n+1} = \frac{1}{2}\frac{e_n^2 f''(y_n)}{f'(x_n)} \approx \boxed{\frac{1}{2}\frac{f''(r)}{f'(r)} e_n^2 = Ce_n^2}$ $\quad |x_{n+1}-r| \le C|x_n - r|^2$

$|e_{n+1}| \le C|e_n|^2$

let $\delta > 0$ s.t. $f'(x) \neq 0$ for $|x-r| \le \delta$ $\quad f'(r)\neq 0$

Set $c(\delta) = \frac{1}{2}\frac{\max_{|x-r|\le\delta}|f''(x)|}{\min_{|x-r|\le\delta}|f'(x)|}$ $\quad$ Note $c(\delta) \to \frac{1}{2}\left|\frac{f''(r)}{f'(r)}\right|$ as $\delta \to 0$

why? Exercise

So $\exists \delta$ s.t. $\delta c(\delta) < 1$. Set $\rho = \delta c(\delta)$

Suppose $|e_0| = |x_0 - r| < \delta$ then $\frac{1}{2}\left|\frac{f''(y_0)}{f'(x_0)}\right| \le c(\delta)$

So $|x_1 - r| = |e_1| \le e_0^2 c(\delta) = |e_0||e_0|c(\delta) \le |e_0|\delta c(\delta) = \rho|e_0| < |e_0| \le \delta$

Thus, the argument can be repeated and

$|e_2| \le c(\delta)e_1^2 \le \rho|e_1| \le \rho^2|e_0|$

$|e_3| \le c(\delta)e_2^2 \le \rho|e_2| \le \rho^3|e_0|$ $\quad (\rho < 1)$

$\vdots$

$|e_n| \le c(\delta)e_{n-1}^2 \le \rho^n|e_0|$ $\quad$ (Clearly $e_n \to 0$)

Theorem on Newton's Method

Let f'' be continuous and r be a simple zero of f (i.e. $f(r)=0$ but $f'(r)\neq 0$). Then $\exists$ a nbd I of r and a constant C s.t. If $x_0 \in I$, then

$x_n \to r$ and $|x_{n+1} - r| \le C(x_n - r)^2$ $\quad (n \ge 0)$

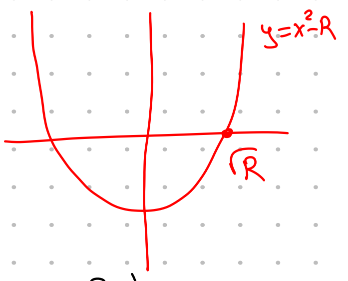
Recall $|e_{n+1}| \le C|e_n|^2$ is called quadratic convergence.

e.g. if $C \approx 1$ and $e_n \approx 0.1 = 10^{-1}$

$e_{n+1} \approx 10^{-2}$ $\quad e_{n+2} \approx 10^{-4}$ $\quad e_{n+3} \approx 10^{-8}$

$e_{n+4} \approx 10^{-16}$ $\approx$ unit roundoff error for 64-bit floating point number

Example Find $\sqrt{R}$ using Newton's method.



let $f(x) = x^2 - R$ then $f(\sqrt{R}) = 0$

$f'(x) = 2x$.

$x_{n+1} = x_n - \frac{x_n^2 - R}{2x_n} = x_n - \frac{x_n}{2} + \frac{R}{2x_n} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$

Say $R = 17$ and $x_0 = 4$

then $x_1 = 4.12$

$x_2 = 4.123106$

$x_3 = 4.1231056256177$

x_4 is correct to 28 figures!