1.3 Difference Equations

Let V be the set of all infinite sequences of $\mathbb{C}$ numbers.

$x, y \in V$ $\quad x_n, y_n \in \mathbb{C}$ for all $n = 1, 2, 3, \dots$

define $x + y \in V$ by $(x+y)_n = x_n + y_n$

for $\lambda \in \mathbb{C}$ $\lambda x \in V$ by $(\lambda x)_n = \lambda x_n$

Then V is a $\mathbb{C}$-vector space.

Define $E : V \to V$ by $(Ex)_n = x_{n+1}$

so $E[x_1, x_2, x_3, \dots] = [x_2, x_3, x_4, \dots]$

E is called the shift operator or displacement operator.

Then $(E^2 x)_n = (EEx)_n = x_{n+2}$

$(E^k x)_n = x_{n+k}$

let $L : V \to V$ in the form $L = \sum_{i=0}^{m} c_i E^i$ $\quad (E^0 = I_V : V \to V$

L : a linear difference operator

$p(\lambda) = \sum_{i=0}^{m} c_i \lambda^i$ is called the characteristic polynomial of L.

Q) Determine $\{x \in V \mid Lx = 0\}$ i.e. the nullspace of L.

"linear difference equation"

e.g. If $L = E - kI$

$$Lx = Ex - kx = [x_2, x_3, x_4, \dots] - [kx_1, kx_2, kx_3, \dots]$$
$$= [x_2 - kx_1, x_3 - kx_2, x_4 - kx_3, \dots]$$

Thus, $Lx = 0 \Rightarrow x_2 = kx_1$, $x_3 = kx_2 = k^2 x_1, \dots, x_n = k^{n-1} x_1$

If $x_1 = 1$, $x = [1, k, k^2, k^3, k^4, \dots]$ and any other solution is a scalar multiple of x. So $\text{null}(L) = \text{span}\{x\}$

Note that $p(\lambda) = \lambda - k$ and $\lambda = k$ is the only root of $p(\lambda)$.

e.g. The following equations describe the same situation:

① $(E^2 - 3E^1 + 2E^0)x = 0$

② $x_{n+2} - 3x_{n+1} + 2x_n = 0 \quad (n \geq 1)$

③ $p(E)x = 0$ $\quad$ where $p(\lambda) = \lambda^2 - 3\lambda + 2$

Say $x = [\lambda, \lambda^2, \lambda^3, \dots]$ i.e. $x_n = \lambda^n$

then ② gives
$$\lambda^{n+2} - 3\lambda^{n+1} + 2\lambda^n = 0$$
$$\lambda^n(\lambda^2 - 3\lambda + 2) = 0$$

so if λ is a root of $p(\lambda)$ then x is a solution of the linear difference equation!

Clearly $p(1) = p(2) = 0$. Thus $x^1 = [1, 1, 1, \dots]$ and $x^2 = [2, 4, 8, 16, \dots]$ are solutions.

Since L is a linear operator, $\text{null}(L)$ is a subspace of V.

Thus, $\lambda_1 x^1 + \lambda_2 x^2 \in \text{null}(L)$ for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

In fact all solutions are of this form.

Theorem Given $L = \sum_{i=0}^{m} c_i E^i$, set $p(\lambda) = \sum_{i=0}^{m} c_i \lambda^i$ and $x(\lambda) = [\lambda^1, \lambda^2, \lambda^3, \dots]$.

If p has only simple (non-repeated) roots, then $\text{null}(L) = \text{span}\{x(\lambda_1), x(\lambda_2), \dots, x(\lambda_m)\}$ where $\lambda_1, \dots, \lambda_m$ are roots of p.

$p(x) = (x - \lambda)^k q(x)$

If a root λ is repeated k times, then each one of the following sequences is also a solution:

$$x(\lambda) = [\lambda, \lambda^2, \lambda^3, \lambda^4, \dots]$$
$$x'(\lambda) = [1, 2\lambda, 3\lambda^2, 4\lambda^3, \dots]$$
$$x''(\lambda) = [0, 2, 6\lambda, 12\lambda^2, \dots]$$
$$\vdots$$
$$x^{(k-1)}(\lambda) = \frac{d^{(k-1)}}{d\lambda^{k-1}} [\lambda, \lambda^2, \lambda^3, \dots]$$

Example Determine the general solution of

$$4x_n + 7x_{n-1} + 2x_{n-2} - x_{n-3} = 0$$

$\lambda = -1 \quad k=2$

$\lambda = \frac{1}{4} \quad k=1$

$$p(\lambda) = 4\lambda^3 + 7\lambda^2 + 2\lambda - 1 = (\lambda+1)^2(4\lambda - 1)$$

so $x(-1) = [-1, 1, -1, 1, \dots]$

$x'(-1) = [1, -2, 3, -4, \dots]$

$x(\tfrac{1}{4}) = [4^{-1}, 4^{-2}, 4^{-3}, \dots]$

$\left(\begin{matrix} [\lambda, \lambda^2, \lambda^3, \dots] \\ [1, 2\lambda, 3\lambda^2, \dots] \end{matrix} \right)$

the general solution is $x = a\,x(-1) + b\,x'(-1) + c\,x(\tfrac{1}{4})$

$$\boxed{x_n = a(-1)^n + b\,n(-1)^{n-1} + c\left(\tfrac{1}{4}\right)^n}$$

HW-Q6 a) Determine a basis for $\text{null}(L)$ where $L = E^2 - E^1 - E^0$

b) let y be the unique element in $\text{null}(L)$ with $y_1 = y_2 = 1$. Express y as a linear combination of the basis vectors you found in part (a).

c) Write a Python func. to compute y_n as in the prev. example. (Purple box).

$x^n = e^{(n \ln x)}$

2.1 Floating-Point Numbers and Roundoff Errors

The decimal system

$$729.504 = 7\cdot 10^2 + 2\cdot 10^1 + 9\cdot 10^0 + 5\cdot 10^{-1} + 0\cdot 10^{-2} + 4\cdot 10^{-3}$$

The binary system

$$(1001.11101)_2 = 1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 + 1\cdot 2^{-1} + 1\cdot 2^{-2} + 1\cdot 2^{-3} + 0\cdot 2^{-4} + 1\cdot 2^{-5}$$

$$= 9.90625$$

We cannot represent every real number if we are allowed to use only finitely many digits.

e.g. $\frac{4}{9} = 0.444\ldots = 0.\bar{4} = 4\cdot 10^{-1} + 4\cdot 10^{-2} + 4\cdot 10^{-3} + \cdots$

similarly $\underbrace{\frac{1}{10}}_{}\!\overset{=0.1}{} = (0.000110011001\ldots)_2 = (0.0\overline{0011})_2$

Recall that if $x \in \mathbb{R}\setminus\mathbb{Q}$, then the digits of x never repeat a pattern.

There are uncountably many numbers that cannot be represented perfectly using the binary (or any) system with finitely many digits.

Rounding

Round the following numbers to 3 decimal places

$0.00102 = \underline{0.001}$ $\quad$ $0.899501 = \underline{0.900}$

$0.781465 = \underline{0.781}$

To round x to n decimal places we look at the $(n+1)^{st}$ digit of x if it is 0, 1, 2, 3, or 4 then the n^{th} digit is not changed and all following digits are discarded. If it is 5, 6, 7, 8, or 9, then the n^{th} digit is increased by 1 and the rest is discarded.

If $\tilde{x}$ is the n-digit approximation to x then

$$|x - \tilde{x}| \le \frac{1}{2}\cdot 10^{-n} \qquad 0.0005 \text{ for } n=3$$

Normalized Scientific Notation

Examples

$$732.5051 = 0.7325051 \cdot 10^3$$

$$-0.005612 = -0.5612 \cdot 10^{-2}$$

for any $0 \ne x \in \mathbb{R}$, there is $\frac{1}{10} \le r < 1$ and $n \in \mathbb{Z}$ s.t. $x = \pm r \cdot 10^n$

Similarly, if $x \ne 0$, then $x = \pm q \cdot 2^m$ where $\frac{1}{2} \le q < 1$ and $m \in \mathbb{Z}$

q: mantissa

m: exponent

For computers we actually choose $\boxed{1 \le q < 2}$, $x = \pm q \cdot 2^m$

Then $q = (1.f)_2$ $\quad$ f: normalized mantissa

Only f is stored as the first digit of q is always 1.

1 bit	8 bits	23 bits
s	Biased exponent (e)	normalized mantissa (f)

total of 32 bits

sign of q

$$s = \begin{cases} 0 & \text{if } x \ge 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$(000\cdots0)_2 = 0$

$(1111\cdots1111)_2 = 255$

$$2^8 - 1 = 255 > e = m + 127 > 0$$

(e = 0, 255 are reserved for special cases like $\pm\infty$, 0, NaN (not a numbers))

f: 23-bit fractional part of x.

Note $127 \ge m \ge -126$

and

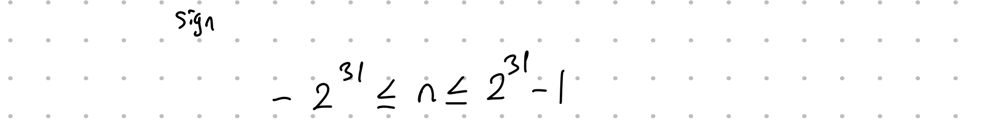
$$3.4\cdot 10^{38} \simeq 2^{128} > \text{machine numbers} \ge 2^{-126} \simeq 1.2\cdot 10^{-38}$$

To use a larger range we use double-precision numbers

i.e. use 64 bits instead of 32.

23 bits for f $\simeq$ 6 decimal places

Numbers that can be represented perfectly:



Integers

32 bits

$$-2^{31} \le n \le 2^{31} - 1$$