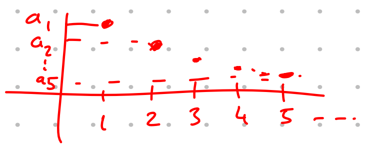
Def^n $f(x) = O(g(x))$ $(x \to \infty)$ means:

$\exists r, C$ s.t. $|f(x)| \le C|g(x)|$ for $x \ge r$.



e.g. $\sqrt{x^2+1} = O(x)$ $(x \to \infty)$

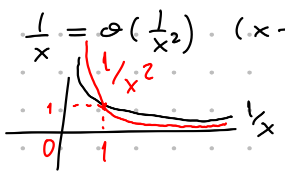
Pf $x^2+1 \le 4x^2$ for $x \ge 2$

so $\sqrt{x^2+1} \le \sqrt{4x^2} = 2x = 2|x|$ for $x \ge 2$ $\square$

$\boxed{\dfrac{\sqrt{x^2+1}}{x} \to 1 \qquad \sqrt{x^2} = x}$ $\qquad \sqrt{x^2+1} \le C|x|$ $\qquad x^2+1 \le Kx^2$

$1 \le (K-1)x^2$

! Note that $\frac{1}{x^2} = o\left(\frac{1}{x}\right)$ $(x \to \infty)$ but $\frac{1}{x} = o\left(\frac{1}{x^2}\right)$ $(x \to 0)$



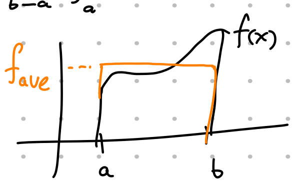
Thm (Mean Value Theorem for Integrals)

Let $u, v \in C^0[a,b]$ and suppose $v \ge 0$. Then, $\exists\, y \in [a,b]$ s.t.

$$\int_a^b u(x)v(x)\,dx = u(y)\int_a^b v(x)\,dx$$

(In particular for $v \equiv 1$, $\frac{1}{b-a}\int_a^b u(x)\,dx = u(y)$)

Pf let $\alpha = \min_{a \le x \le b} u(x)$ $\quad \beta = \max_{a \le x \le b} u(x)$.



Then $\alpha \le u(x) \le \beta$ for $a \le x \le b$.

$$\alpha v(x) \le u(x)v(x) \le \beta v(x)$$

so $\alpha \int_a^b v(x)\,dx \le \int_a^b u(x)v(x)\,dx \le \beta \int_a^b v(x)\,dx$. let $I = \int_a^b v(x)\,dx$.

Note that $I = 0$ iff $v \equiv 0$ on $[a,b]$.

If $v \equiv 0$, then the equality clearly holds.

Otherwise, $I \ne 0$.

$$I\alpha \le \int_a^b u(x)v(x)\,dx \le I\beta$$

$$\alpha \le \frac{1}{I}\int_a^b u(x)v(x)\,dx \le \beta \qquad \text{By I.V.T., } \exists\, y \in [a,b]$$

s.t. $u(y) = \frac{1}{I}\int_a^b u(x)v(x)\,dx. \rightsquigarrow \int_a^b u(x)v(x)\,dx = u(y)\,I = u(y)\int_a^b v(x)\,dx$ $\square$

Nested Multiplication

How to compute x^n?

If we use $x^n = (\cdots((xx)x)x\cdots)x$ we do $n-1$ multiplications

If $n = 2^k$, we only need $k = \log_2 n$ multiplications!

$$x \to x^2 \to x^{(2^2)} \to x^{2^3} \to \cdots \to x^{2^k} = x^n.$$

\# of mult. opers: n $\quad n-1$ $\quad n-2$ $\;\cdots\;$ 1 $\quad$ 0 $\qquad 1+2+3+\cdots+n = O(n^2) = \frac{n(n+1)}{2}$

Note that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$

\# of mult. opers: 1 $\quad$ 1 $\quad$ 1 $\quad$ 1 $\qquad n = O(n)$

$= a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + x(a_n))))$

but the second line involves only n multiplications and n additions.

Pseudo-Code: (Horner's method)

```
p ← a_n
for k=n-1 to 0 step -1 do{
    p ← xp + a_k
}end do
```

$p \leftarrow a_n$ $\qquad\qquad\qquad\qquad p = a_n$

$k = n-1$ $\quad p \leftarrow xp + a_{n-1}$ $\qquad p = a_{n-1} + xa_n$

$k = n-2$ $\quad p \leftarrow xp + a_{n-2}$ $\qquad p = a_{n-2} + x(a_{n-1} + xa_n)$

$\vdots$

$k = 0$ $\quad p \leftarrow xp + a_0$

Exercise: Write a Python function that takes a list of coefficients $[a_0, a_1, \ldots, a_n]$ and a x value to compute $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ using Horner's method.

Upper and Lower Bounds

Def^n Let $S \subset \mathbb{R}$. We say that S is bounded if $\exists\, a, b \in \mathbb{R}$ s.t.

$a \le x \le b$, for all $x \in S$. $\quad a$: lower bound $\quad b$: upper bound

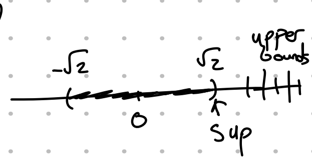
Axiom: Any nonempty set of real numbers that has an upper bound has a least upper bound (supremum).

Def^n The supremum of S is v ($v = \sup S = \mathrm{lub}\, S$) iff

1. v is an upper bound for S and
2. no real number smaller than v is an upper bound for S.

Example What is $\sup S$ if $S = \{x \mid x^2 < 2\}$?

$S = (-\sqrt{2}, \sqrt{2})$ $\qquad \sup S = \sqrt{2}$

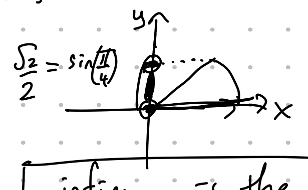


For $f: A \to \mathbb{R}$, $\quad \sup_{x \in A} f(x) = \sup\{f(x) \mid x \in A\}$

$\sup(\mathbb{R}) = \infty$
$\inf(\mathbb{R}) = -\infty$

e.g. $\sup_{0 < x < \frac{\pi}{4}} (\sin(x)) = \frac{\sqrt{2}}{2}$

$\sin\left(\left(0, \frac{\pi}{4}\right)\right) = \left(0, \frac{\sqrt{2}}{2}\right)$



infimum is the greatest lower bound defined similarly

1.3 Difference Equations

Let V be the set of all infinite sequences of $\mathbb{C}$ numbers.

$x, y \in V$ $\qquad x_n, y_n \in \mathbb{C}$ for all $n = 1, 2, 3, \ldots$

define $x + y \in V$ by $(x+y)_n = x_n + y_n$

for $\lambda \in \mathbb{C}$ $\lambda x \in V$ by $(\lambda x)_n = \lambda x_n$

Then V is a $\mathbb{C}$-vector space.

Define $E: V \to V$ by $(Ex)_n = x_{n+1}$

so $E[x_1, x_2, x_3, \ldots] = [x_2, x_3, x_4, \ldots]$

E is called the shift operator or displacement operator.

Then $(E^2x)_n = (EEx)_n = x_{n+2}$

$(E^kx)_n = x_{n+k}$