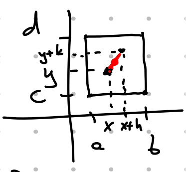
Thm (Taylor's thm in 2 variables)

Say $f \in C^{n+1}([a,b]\times[c,d])$ and $(x,y), (x+h, y+k) \in [a,b]\times[c,d]$

then

$$f(x+h, y+k) = \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x,y) + E_n(h,k)$$



where

$$E_n(h,k) = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x+\theta h, y+\theta k)$$

for $0 \le \theta \le 1$

$\left(f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y\,\partial x}\right)$

(notation: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$)

Note $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^0 f(x,y) = f(x,y)$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^1 f(x,y) = hf_x(x,y) + kf_y(x,y)$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2 f_{xx}(x,y) + 2hk\, f_{xy}(x,y) + k^2 f_{yy}(x,y)$$

Example Find the first few terms in Taylor series expansion for $f(x,y) = \cos(xy)$.

$f(x,y) = \cos(xy)$ $\quad f_x(x,y) = -\sin(xy)y$ $\quad f_y(x,y) = -\sin(xy)x$

$f_{xx}(x,y) = -\cos(xy)y^2$ $\quad f_{yy}(x,y) = -\cos(xy)x^2$

$f_{xy}(x,y) = -\cos(xy)xy - \sin(xy)$

$$f(x+h, y+k) = f(x,y) + hf_x(x,y) + kf_y(x,y) + \frac{1}{2}\left(h^2 f_{xx}(x,y) + 2hk f_{xy}(x,y) + k^2 f_{yy}(x,y)\right) + \cdots + E_n(h,k)$$

$n=1$

$$\cos(x+h, y+k) = \cos(xy) - hy\sin(xy) - kx\sin(xy) + E_1(h,k)$$

where $E_1(h,k) = \frac{1}{2}\Big\{-h^2(y+\theta k)^2\cos((x+\theta h)(y+\theta h)) - 2hk\big[\cos((x+\theta h)(y+\theta k))(x+\theta h)(y+\theta k) - \sin((x+\theta h)(y+\theta k))\big] - k^2(x+\theta h)^2\cos((x+\theta h)(y+\theta k))\Big\}$

1.2 Orders of Convergence and Additional Basic Concepts

Defn Say $x_1, x_2, x_3 \dots \in \mathbb{R}$. We write $\lim_{n\to\infty} x_n = L$ or $x_n \to L$ (as $n\to\infty$) if for any $\varepsilon > 0$, there exists a $N \in \mathbb{R}$ s.t. $|x_n - L| < \varepsilon$ whenever $n > N$.

Example $\lim_{n\to\infty} \frac{n+1}{n} = 1$

Pf Given $\varepsilon > 0$, we choose $N = \frac{1}{\varepsilon}$ so that "if $n > N$, then $|\frac{n+1}{n} - 1| < \varepsilon$."

$$\varepsilon > \left|\frac{n+1}{n} - 1\right| = \left|\frac{n+1}{n} - \frac{n}{n}\right| = \left|\frac{1}{n}\right| = \frac{1}{n}$$

So $n > \frac{1}{\varepsilon} = N$

Orders of Convergence

let $\{x_n\}$ be a sequence of real numbers with $x_n \to L$ as $n \to \infty$. We say that

- the rate of convergence is at least linear if $\exists\, c < 1$ and $N \in \mathbb{Z}$ s.t. $|x_{n+1} - L| \le c|x_n - L|$ $(n \ge N)$ "there exists"
- the rate of convergence is at least super linear if $\exists \{\varepsilon_n\} \to 0$ and $N \in \mathbb{Z}$ s.t. $|x_{n+1} - L| \le \varepsilon_n |x_n - L|$ $(n \ge N)$
- the rate of convergence is at least quadratic if $\exists\, C \in \mathbb{R}^+$ and $N \in \mathbb{Z}$ s.t. $|x_{n+1} - L| \le C|x_n - L|^2$ $(n \ge N)$

In general;

- the rate of convergence is at least of order α if $\exists\, C, \alpha \in \mathbb{R}^+$ and $N \in \mathbb{Z}$ s.t. $|x_{n+1} - L| \le C|x_n - L|^\alpha$ $(n \ge N)$

Example Since $(1 + \frac{1}{n})^n \to e$, we can use $x_n = (1+\frac{1}{n})^n$ to approximate e. However, numerical evidence suggests that

$$\frac{|x_{n+1} - e|}{|x_n - e|} \to 1$$

so the convergence is worse than linear! as we can tell from the following values:

$x_1 = 2$, $x_{10} = 2.593742$, $x_{30} = 2.674319$ $x_{50} = 2.691588$

$x_{1000} = 2.716924$ $\quad e = 2.7182818\dots$

Example Set $\begin{cases} x_1 = 2 \\ x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n} \end{cases}$ then $x_n \to \sqrt{2}$

and

$x_1 = 2$, $x_2 = 1.5$, $x_3 = 1.4166667$ $x_4 = 1.414216$

$\sqrt{2} = 1.414213562\dots$

$$\frac{|x_{n+1} - \sqrt{2}|}{|x_n - \sqrt{2}|^2} \le 0.36$$

at least quadratic

Big O and Little o Notation

let $\{x_n\}$ and $\{\alpha_n\}$ be sequences. We write

- $x_n = O(\alpha_n)$ if $\exists\, C, n_0$ s.t. $|x_n| \le C|\alpha_n|$ when $n \ge n_0$.

x_n is "big oh" of α_n

"If $\alpha_n \to 0$ then $x_n \to 0$ at least as rapidly as α_n".

- $x_n = o(\alpha_n)$ if $\exists \{\varepsilon_n\} \to 0$ s.t. $|x_n| \le \varepsilon_n |\alpha_n|$

"If $\alpha_n \to 0$ then $x_n \to 0$ more rapidly than α_n".

$\to$ this is roughly saying that $\dfrac{x_n}{\alpha_n} \to 0$ $\qquad \dfrac{\left(\frac{n+1}{n^2}\right)}{\frac{1}{n}} \to C \neq 0$

e.g. $\dfrac{n+1}{n^2} = O\left(\dfrac{1}{n}\right)$ $\qquad \dfrac{1}{n \ln n} = o\left(\dfrac{1}{n}\right)$

$\dfrac{1}{n} = o\left(\dfrac{1}{\ln n}\right)$ $\qquad \dfrac{5}{n} + e^{-n} = O\left(\dfrac{1}{n}\right)$

Recall that $\ln 2 \simeq \sum_{k=1}^{n} (-1)^{k+1} \dfrac{1}{k}$. In other words,

$x_n = \ln 2 - \sum_{k=1}^{n} (-1)^{k+1} \dfrac{1}{k} \to 0$. $\quad x_n = O\left(\dfrac{1}{n}\right)$ very slow convergence!

$|E_n(x)| < \dfrac{1}{n+1}$

On the other hand,

$$e^x - \sum_{k=0}^{n} \frac{1}{k!} x^k = O\left(\frac{1}{n!}\right) \quad \text{for } |x| \le 1$$

this is very rapid convergence.

We also use O, o notations for functions of $\mathbb{R}$.

e.g. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} \cdots$

then $\sin x = x - \dfrac{x^3}{3!} + O(x^5)$ $\quad$ as $x \to 0$

meaning: $\left| \sin x - x + \dfrac{x^3}{3!} \right| \le C |x^5|$ $\quad$ for some $C > 0$ and $\delta > 0$ whenever $|x| < \delta$

Exercise Prove the above statement using Taylor's theorem with $n=4$.

Q) Which one of the following is correct (as $x \to 0$)?

$\sin x = x - \dfrac{x^3}{3!} + O(x^4)$ $\quad$ or $\quad$ $\cancel{\sin x = x - \dfrac{x^3}{3!} + O(x^6)}$