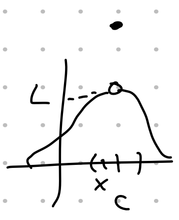
Office Hours on Wednesdays 9:30 – 11:00 am or by appointment.

A total of 7 HW sets. Some will contain programming questions.

1.1 Basic Concepts and Taylor's Theorem

Def^n Say $f:\mathbb{R}\to\mathbb{R}$ then $\lim\limits_{x\to c} f(x) = L$ if for any $\varepsilon>0$, there exists a $\delta>0$ s.t.



(✱) $\quad |f(x) - L| < \varepsilon$ whenever $0<|x-c|<\delta$

If there is no number L with this property, the limit of f at c does not exist (D.N.E.).

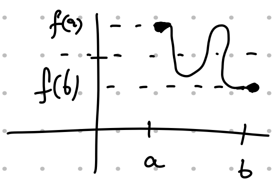
Warm up Exercise a) Prove that $\lim\limits_{x\to 2} x^2 = 4$ (i.e. find a δ in terms of ε s.t. (✱) is true.)

b) Prove that $\lim\limits_{x\to 0} \frac{|x|}{x}$ D.N.E.

The function f is said to be continuous at c if
$$\lim_{x\to c} f(x) = f(c)$$

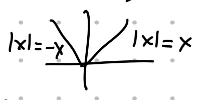
Thm (I.V.T.)

On an interval $[a,b]$, a cont. fn. assumes all values between $f(a)$ and $f(b)$.



Def^n The derivative of f at c (if it exists) is defined by the equation
$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

e.g. for $f(x) = |x|$, $f'(c) = \frac{|c|}{c}$ if $c\neq 0$ and DNE if $c=0$.



$|x|=-x$ $\quad |x|=x$

$f'(x) = \begin{cases} +1 & \text{if } x>0 \\ -1 & \text{if } x<0\end{cases}$

Notation $C(\mathbb{R})$ = the set of all cont. fns $f:\mathbb{R}\to\mathbb{R}$.

$C^1(\mathbb{R})$ = the set of all fns $f:\mathbb{R}\to\mathbb{R}$ s.t. $f':\mathbb{R}\to\mathbb{R}$ is cont.

$C^2(\mathbb{R})$ = $\quad$ // $\quad$ // $\quad f'':\mathbb{R}\to\mathbb{R}$ //

etc.

We also set $C(\mathbb{R}) = C^0(\mathbb{R})$.

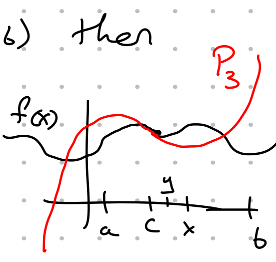
$C^\infty(\mathbb{R})$ is the intersection of all $C^n(\mathbb{R})$.

$e^x,\ \sin x,\ x^5+4x \in C^\infty(\mathbb{R})$

Similarly $C^n[a,b] = \{ f:[a,b]\to\mathbb{R} \mid f^{(n)}:[a,b]\to\mathbb{R} \text{ is Cont}\}$

Thm (Taylor's Theorem with Lagrange Remainder)

If $f\in C^n[a,b]$ and if $f^{(n+1)}$ exists on (a,b) then for $x, c\in[a,b]$,



(✱✱) $$f(x) = \underbrace{\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(x-c)^k}_{P_n} + E_n(x)$$

where $$E_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(y)(x-c)^{n+1}$$ for some y between x and c.

Some well known Taylor series

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad x\in\mathbb{R}$$

$$\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \qquad \sin x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \qquad x\in\mathbb{R}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty}x^k \qquad -1<x<1$$

Note that in the RHS of (✱✱) the first term is a polynomial. It is a simple function that approximates $f(x)$ with error term $E_n(x)$.

Example a) Find the degree n Taylor polynomial of $\ln x$ centered at $c=1$.

$f(x) = \ln x \qquad f(1) = \ln 1 = 0$

$f'(x) = x^{-1} \qquad f'(1) = 1$

$f''(x) = -x^{-2} \qquad f''(1) = -1$

$f'''(x) = +2x^{-3} \qquad f'''(1) = 2$

$\vdots$

$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,x^{-k} \qquad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

Thus, $$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=1}^{n}\frac{(-1)^{k+1}\cancel{(k-1)!}}{\cancel{k!}_{\,k}}(x-1)^k$$

$$= \sum_{k=1}^{n}\frac{(-1)^{k+1}(x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n+1}(x-1)^n}{n}$$

b) How many terms do we need to use to compute $\ln(2)$ with accuracy of 10^{-8}?

Exercise: Do the same thing for $\ln(1.5)$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots + (-1)^{n+1}\frac{1}{n} + E_n(2)$$

$$|E_n(2)| = \left|\frac{1}{(n+1)!}f^{(n+1)}(y)(2-1)^{n+1}\right| = \frac{1}{n+1}y^{-(n+1)} < \frac{1}{n+1}$$ since $c=1<y<x=2$

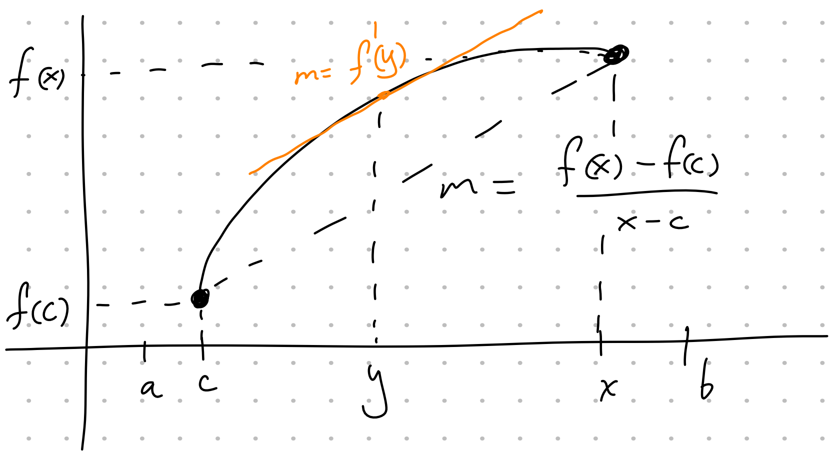
So we want $\frac{1}{n+1} \le 10^{-8}$ or $n+1 \ge 10^{8}$

So this is not very practical!

Thm (M.V.T.) (Taylor's thm with $n=0$)

If $f \in C[a,b]$ and if f' exists on (a,b) then for $x, c \in [a,b]$, $f(x) = f(c) + f'(y)(x-c)$ where y is between c and x.

or $$f'(y) = \frac{f(x)-f(c)}{x-c} \quad (\text{if } x \ne c)$$



Thm (Taylor's Thm with Integral Remainder)

If $f \in C^{n+1}[a,b]$, then for $x, c \in [a,b]$

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + R_n(x)$$

where $$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Pf (for $n=2$)

$$R_2(x) = \frac{1}{2}\int_c^x f'''(t)(x-t)^2\,dt$$ (we use integration by parts $u = \frac{(x-t)^2}{2}$ $dv = f'''(t)dt$, $du = -(x-t)dt$ $v = f''(t)$)

$$R_2(x) = \frac{1}{2} f''(t)(x-t)^2\Big|_c^x + \underbrace{\boxed{\int_c^x f''(t)(x-t)\,dt}}_{R_1(x)}$$ (we use integration by parts again)

$$R_2(x) = -\frac{1}{2} f''(c)(x-c)^2 + \left[f'(t)(x-t)\Big|_c^x + \int_c^x f'(t)\,dt \right].$$

$$R_2(x) = -\frac{1}{2} f''(c)(x-c)^2 - f'(c)(x-c) + f(x) - f(c)$$

So $$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2} f''(c)(x-c)^2 + R_2(x)$$

Thm (Alternative form of Taylor's Theorem) $\begin{pmatrix} x, & c \\ \updownarrow & \updownarrow \\ x+h & x \end{pmatrix}$

$$f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) + E_n(h)$$

where $E_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(y)$ for y between x and $x+h$

Example Determine Taylor's formula for A^{x+h} and approximate $10^{1.0001}$.

Let $f(x) = A^x$ then $f'(x) = A^x \ln A$

and $f^{(n)}(x) = A^x (\ln A)^n$

$$A^{x+h} = \sum_{k=0}^{n} \frac{h^k}{k!} A^x (\ln A)^k + E_n(h)$$
$$= A^x \left(\sum_{k=0}^{n} \frac{h^k}{k!} (\ln A)^k \right) + E_n(h)$$

So for $A = 10$, $x = 1$, $h = 0.0001 = 10^{-4}$,

$$10^{1.0001} \approx 10\left(1 + 10^{-4}(\ln 10) + \frac{10^{-8}}{2}(\ln 10)^2 + \cdots\right)$$

$$\approx 10.0023000265\ldots$$